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ELEMENTS OF THE APPLIED THEORY OF ELASTIC VIBRATION

MIR PUBLISHERS

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ОСНОВЫ ПРИКЛАДНОЙ ТЕОРИИ УПРУГИХ КОЛЕБАНИЙ

ИЗДАТЕЛЬСТВО «МАШИНОСТРОЕНИЕ»
МОСКВА

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ELEMENTS OF THE APPLIED THEORY OF ELASTIC VIBRATION

Translated from the Russian
by
M. Konyaeva

MIR PUBLISHERS
MOSCOW 1971

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NOTATION

A = amplitude, area
$A_1, A_2, \ldots$ = coefficients
a = amplitude, constant, distance
a, b = coefficients, dimensions
$B_1, B_2, \ldots$ = coefficients, constants of integration
$b_1, b_2, \ldots$ = coefficients
$C_1, C_2, \ldots$ = constants of integration
c = damping constant
c_* = equivalent damping constant
$c_1, c_2, \ldots$ = coefficients
D = flexural rigidity of plate
$D_1, D_2, \ldots$ = constants of integration
D, d = diameter
$d_1, d_2, \ldots$ = coefficients
E = modulus of elasticity
EA = tensile stiffness
EI = flexural rigidity of beam
e = eccentricity, exponential constant (2.71828...)
F = force
f = deflection
f_s = static displacement
G = shear modulus
GI_p = torsional rigidity
g = acceleration of gravity
h = height, thickness
I = moment of inertia of an area
I_p = polar moment of inertia of an area
i = gear ratio
J = moment of inertia of mass
k = constant in general or spring constant in particular
L, l = length
M = moment or torque
M_b = bending moment
m = mass
N = longitudinal force, number of cycles, power
n = constant, damping factor, number in general, number of cycles per second in particular
P = force
P_{cr} = critical force
p = natural circular frequency
p_* = natural circular frequency of damped vibration
Q = generalized force, shearing force, weight
q = generalized co-ordinate, load per unit length, pressure
R = resisting force in general or dry friction force in particular

R, r = radius
r = deflection
r, θ = polar co-ordinates
S = impulse
T = friction force, kinetic energy, period of vibration
t = time
U = potential energy
u = longitudinal displacement, radial displacement
v = deflection, velocity
v_0 = initial velocity
W = weight, work
w = deflection
x = displacement
x_0 = initial displacement
x_{st} = static displacement
x, y, z = rectangular co-ordinates
y = displacement
y_0 = initial displacement
α (alpha) = angular acceleration, phase angle or some other angle
β (beta) = repetition factor
γ (gamma) = phase angle or some other angle, shear angle, specific weight
δ (delta) = logarithmic decrement
δ_{ik} = unit displacement or influence coefficient
ε (epsilon) = angle, strain
ε_r = strain in radial direction
ε_0 = strain in circumferential direction
θ (theta) = angle, angular displacement
$\varkappa$ (kappa) = amplitude ratio, correction factor
λ(lambda) = characteristic number
μ (mu) = amplification (magnification) factor, Poisson's ratio
μ_* = transmissibility
π (pi) = 3.14159...
ρ (rho) = mass density, radius of curvature, radius of gyration
σ (sigma) = stress
σ_r = radial stress
σ_θ = circumferential stress
τ (tau) = shearing stress, time
φ (phi) = angle of rotation or angular displacement
Ψ (psi) = energy dissipated per cycle
ψ (psi) = angle
ω (omega) = angular velocity, circular frequency
ω_{cr} = critical speed

INTRODUCTION

The periodic nature of operation of most machines determines the periodicity of loading and deformation of both their individual members and those structures which serve as supports or foundations; it may be said that elastic vibrations accompany the operation of each machine.

In some cases, however, vibrations occur in the absence of periodic excitation. These are, for instance, relatively simple processes of free vibration developing upon a sudden disturbance of the state of equilibrium of a mechanical system as well as more complicated and, at the same time, less studied processes such as self-excited vibrations.

It is difficult to indicate a domain of engineering in which the study of elastic vibrations would not be an urgent problem. Much attention is given by investigators to vibration of structures of widely differing purposes: turbine rotors, internal-combustion engine shafts, turbine blades, propellers, automobiles and railroad cars, ships and aircraft, engineering structures, industrial-building floors, parts worked on metal-cutting machines, jigging conveyors, etc.

In certain cases vibrations impede the normal service or even directly endanger the strength by gradually promoting fatigue failure; in such cases the theory may indicate ways of reducing detrimental vibrations. At the same time it enables one to substantiate and optimize the manufacturing processes which use vibrations purposefully (as in jigging conveyors).

For all the variety of problems treated in the theory of elastic vibration there is a deep intrinsic connection between outwardly different problems. The existence of common laws forms the fundamental basis for the general theory which enables one to consider simultaneously wide classes of phenomena covering a host of particular problems.

We may distinguish at least the following five sufficiently independent categories of vibratory processes differing in their nature:

free vibrations, i.e., vibrations which are performed by a mechanical system having no energy supply from outside if the system is disturbed from its position of equilibrium and then released;

critical states of rotating shafts and rotors which consist in a sudden increase in the deflections of their axes at definite speeds of rotation (or in definite ranges of speeds);

forced vibrations which result when the mechanical system is acted on by fluctuating external forces (driving forces);

parametric vibrations caused by periodic variations of the parameters of a system (for example, its stiffness);

self-excited vibrations, i.e., vibratory processes which are maintained by constant sources of energy of a nonvibratory nature.

Each of these categories of vibratory processes is discussed in the appropriate chapter.

CHAPTER I

FUNDAMENTALS

1. Number of Degrees of Freedom of an Elastic System

The complexity of a theoretical analysis of vibration depends largely on the number of degrees of freedom of the mechanical system in question. The *number of degrees of freedom* of a mechanical system is defined as the number of independent co-ordinates which determine uniquely the positions of all particles of the system.

In dynamic problems, particularly in vibration problems, the positions of the particles of a system vary with time so that the above co-ordinates are functions of time. The fundamental problem of dynamic analysis is to find these functions, i.e., to determine the motion of the system. It is an easy matter then to find strains, stresses and internal forces in the constraints of the system.

Every mechanical system involves an infinite number of particles and consequently the number of degrees of freedom is always infinite. However, in solving practical problems use is generally made of simplified schemes which are characterized by a finite number of degrees of freedom. In such design schemes some (the lightest) parts of the system are assumed to be massless and are represented as deformable inertialess constraints; the bodies for which the property of inertia is retained in the design scheme are then considered to be particles ("concentrated masses") or absolutely rigid bodies.

In an endeavour to simplify the design scheme one should remember that the neglect of all inertia properties of a given system deprives the problem of dynamic specific features.

Consider, for instance, a massless spring (Fig. 1*a*) with a force $P(t)$, given as a function of time, applied to its end. If k is the stiffness of the spring, the displacement x of its end is defined by the ordinary static formula

$$x = \frac{P}{k}. \tag{1.1}$$

This formulation of the problem is not really dynamic though the displacement thus found is not constant but represents a function of time. The true dynamics of processes in real mechanical systems is associated with *the property of inertia* and this property must be reflected in one way or another in the design scheme.

A simple example of a dynamic system with one degree of freedom is represented in Fig. 1*b*. Here we can no longer work with purely static relationships; thus, we observe that the reaction R of the spring is not equal to the external force P.

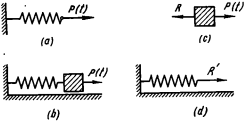


Fig. 1

The differential equation of motion of the mass in the x direction (Fig. 1*c*) is of the form

$$P+R_x=m\ddot{x}, \tag{1.2}$$

where $R_x=-kx$ is the projection of the reaction of the spring on the x axis. Thus we obtain

$$m\ddot{x}+kx=P(t). \tag{1.3}$$

Unlike expression (1.1) which serves to calculate x directly, relation (1.3) represents a *differential equation* in the function x. To find the form of this function it is necessary to integrate the differential equation (1.3). After solving Eq. (1.3) the function $x=x(t)$ is used to find internal forces, stresses, etc.

It may be said that in the above example the function x alone defines completely all the elements of the state of strain at any instant. Such systems possess *one degree of freedom*.

The systems shown in Fig. 2 fall into this type. The characteristic co-ordinate for the diagram of Fig. 2*a* is the ordinate y of the mass, while for the diagram of Fig. 2*b* it is the angle of rotation φ of the rigid body (in both cases the elastic constraints are assumed to be massless).

The systems represented in Fig. 2*c*, *d*, *e*, *f* each have *two degrees of freedom*. In the diagram of Fig. 2*c* there are two concentrated masses and the motion of the system is completely determined by two func-

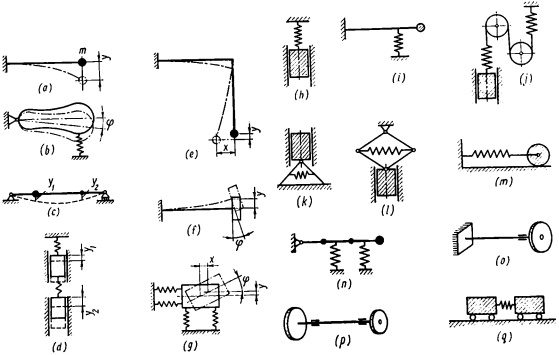


Fig. 2

tions

$$y_1 = y_1(t), \quad y_2 = y_2(t).$$

The same is true for the diagram of Fig. 2*d*.

For the two-dimensional system of Fig. 2*e* it is necessary to take into account that the point mass is capable of motion in two directions and it is natural to choose the co-ordinates as

$$x = x(t), \quad y = y(t).$$

A feature of the diagram shown in Fig. 2*f* is that the dimensions of the mass attached to the beam are finite; in this case it is necessary to take into account the rotary inertia of the mass as during vibration the beam will be loaded at its end not only by the inertia force of the mass but also by the moment of the inertia force. The motion of the system is defined by the displacement $y = y(t)$ and the angle of rotation $\varphi = \varphi(t)$.

The system represented in Fig. 2*g* has *three degrees of freedom* and its motion is defined by the functions

$$x = x(t), \quad y = y(t), \quad \varphi = \varphi(t).$$

In the diagrams of Fig. 2*h* to *l* the masses are suspended in different ways but are capable of executing vibrations along a single fixed straight line; therefore, regardless of the differences in the arrangement of the elastic constraints each of these systems has one degree of freedom.

The system shown in Fig. 2*m* has one degree of freedom if rolling is not accompanied by slipping. Figure 2*n* shows a perfectly rigid beam whose position at any instant is determined by a single quantity—the angle of rotation about an immovable hinge support; regardless of the number of masses and springs this system also has just one degree of freedom. The system shown in Fig. 2*o* can execute torsional vibrations about the shaft axis and hence is basically the same as the system represented in Fig. 2*b*; if only the mass of the disk is taken into account, the motion of the system is completely determined by the function $\varphi(t)$.

The system presented in Fig. 2*p* has, strictly speaking, not one but two degrees of freedom since its motion is described by two functions, $\varphi_1(t)$ and $\varphi_2(t)$, which express the angles of rotation of the disks about the longitudinal axis of the system. However, the elastic vibrations are defined by just one function—the relative angle of rotation of the disks, $\varphi = \varphi_2 - \varphi_1$; in this sense the system has only one (vibratory) degree of freedom. The same feature is inherent in the system shown in Fig. 2*q*.

In all the examples above the number of degrees of freedom is finite (and, besides, not large) due to the usual simplifications: it was assumed that the deformable parts of a system were massless while the bodies possessing mass were absolutely non-deformable. In actual fact the properties of deformability and inertia always accompany each other, hence every mechanical system, as already stated, has *an infinite number of degrees of freedom*.

Sometimes it is recognized that the properties of deformability and inertia are distributed throughout the volume of a system but by *specifying a priori* (and arbitrarily to a certain extent) *the configuration of the system during vibration* (*the mode of vibration*) we form a design scheme which has a finite number of degrees of freedom.

Thus, if we assume that the deflection curve of a vibrating simply supported beam with distributed mass is a sine curve (Fig. 3)

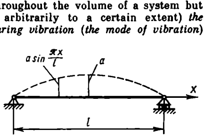


Fig. 3

$$y(x, t) = a\sin\frac{\pi x}{l},$$

the configuration of the system at any instant is completely determined by a single quantity—the deflection $a = a(t)$ at mid-length; the system has only one degree of freedom.

If a more complex representation of the deflection curve is assumed for the same beam

$$y(x, t) = a_1\sin\frac{\pi x}{l} + a_2\sin\frac{2\pi x}{l},$$

the position of the system at any instant is determined by the functions $a_1(t)$, and $a_2(t)$, i.e., the system has two degrees of freedom.

It should be remembered that although the accuracy of dynamic analysis may be improved by adding "recognized" degrees of freedom, it is generally sufficient for practical purposes to consider a small number of the most important degrees of freedom.

In conclusion we note that in some cases it is possible to obtain the exact solution of vibration problems for elastic systems without recourse to any simplifications whatsoever, i.e., to allow for the actual infinity of the number of degrees of freedom; true, this can be done only for systems of simple structure, such as straight beams of constant section and uniform mass distribution.

2. Classification of Forces

The forces acting on a mechanical system from outside and also the internal forces developed in its constraints are widely different both in their nature and in the part they play in the vibratory process.

Restoring Forces. These forces occur when the system is displaced from its equilibrium position and tend to restore the system to that position. It may be said that the vibratory properties of mechanical systems result from the action of these very forces.

The basic type of restoring forces are *elastic forces*. In the simplest case of a linearly deformable system the elastic restoring force is proportional to the deflection of the system; the property of the elastic constraints is then completely determined by a single number—the spring constant. The *spring constant* or *spring stiffness* k is the factor of proportionality between the force P acting statically on the system and the displacement it produces; if this displacement is denoted by y, then

$$P = ky.$$

However, not infrequently the relation between the force and the displacement is not linear and the elastic properties of the constraints cannot be determined by a single number. In such cases the above properties are determined by a *spring characteristic* whose equation

$$P = P(y)$$

is usually plotted in the co-ordinate system y, P. The spring characteristic is plotted (by calculation or experimentally) for static-loading conditions of the system.

Figure 4 presents several simple systems and the relevant spring characteristics.

A distinction is made between *soft* and *stiff* non-linear spring characteristics. Soft characteristics are those with gradually decreasing inclination (Fig. 4*a*) and stiff characteristics have gradually increasing inclination (Fig. 4*b*).

In some constructions the spring characteristics change abruptly or have discontinuities. Thus, the spring characteristic of the mechanical system represented in Fig. 4*c* is bilinear, and part of the spring characteristic of the mechanical system shown in Fig. 4*d* (system with a clearance) coincides with the axis of abscissas. To the system with an initial compression (Fig. 4*e*) corresponds a discontinuous spring characteristic.

For systems whose deflection is characterized by an angular displacement φ, the spring characteristic represents the relation between the moment M acting statically on the system and the above displacement (Fig. 5). The spring characteristic may be of the form of any one of the graphs shown in Fig. 4 (with the axes y and P replaced by the axes φ and M, respectively) depending on the properties of the elastic constraints.

A *buoyant force*, as well as elastic forces, possesses restoring properties. Figure 6 represents a floating body (float, pontoon, vessel), the solid line indicating the equilibrium position and the dashed line the displaced position. If the vertical displacement is of a purely translational nature (Fig. 6a), a vertical restoring force is developed; when the body is inclined (Fig. 6b), a restoring couple (moment) comes into action. The linearity or non-linearity of the characteristic depends on the outline of the walls of the body. Thus, for vertical displacements y of a wall-sided body the characteristic is linear (see Fig. 6a). When angular displacements, or vertical displacements of a body which is not wall-sided, are involved, the characteristic is a nonlinear and soft one (see Fig. 6b).

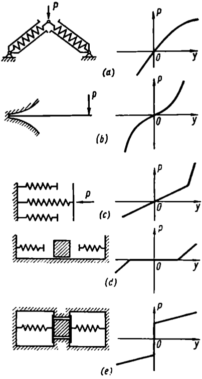


Fig. 4

The restoring action may be exerted by a *gravity force* such as that of a liquid in connecting vessels (Fig. 7a). *The moment of a gravity force* often possesses restoring properties; a simple example is provided by a *pendulum* (Fig. 7b). The deflections of the system are determined by the angle φ while the measure of the restoring action is the moment of the gravity force about the hinge axis. The characteristic of this system is of the same form as the graph in Fig. 6b.

In some cases restoring forces are of the mixed type: thus, for the pendulum in Fig. 7c the restoring action is exerted both by the weight and the elastic force.

The characteristics of the systems represented in Figs. 6 and 7 are often referred to as *quasi-elastic* emphasizing thereby the similarity of all types of restoring forces to the elastic force.

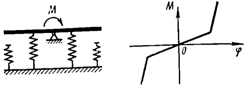


Fig. 5

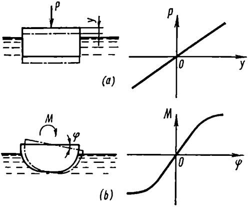


Fig. 6

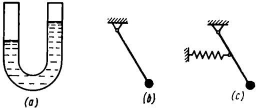
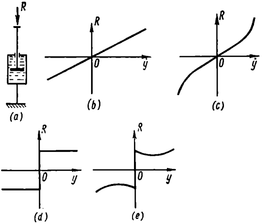


Fig. 7

Dissipative Forces. During vibration, friction forces inevitably develop in mechanical systems as well as restoring forces. They do irrecoverable work, and the result is a dissipation of mechanical energy; accordingly such forces are called *dissipative forces.* Among these are friction forces at supports and connections of a mechanical system, the resistance of a medium (either fluid or gaseous) in which vibrations take place, internal friction in the material of the elements of a system, and, finally, forces which are produced in energy absorbers (dashpots).



(a) (b) (c) (d) (e)

Fig. 8

Whatever the nature of friction, the direction of dissipative forces at any instant during the motion is opposite to the *velocity*, the magnitude of the force being related in one way or another to the magnitude of the velocity v.

Dissipative properties are described by means of *friction characteristics.* Thus, Fig. 8*a* shows the simplest friction element—a viscous dashpot whose resistance is proportional to the velocity of the piston along the cylinder and Fig. 8*b* represents the linear characteristic of the dashpot, i.e., the graph of the relation

$$R = c\dot{y},$$

where R implies *the force applied to the dashpot.*

In some cases the friction characteristic may be nonlinear (Fig. 8*c*), and in the case of dry friction it is discontinuous (Fig. 8*d* and *e*). Figure 8*d* shows the friction characteristic corresponding to the elementary Amontons-Coulomb law according to which the magnitude of the friction force is independent of the velocity. Figure 8e presents an improved dry friction characteristic.

Driving Forces. The characteristics of restoring and dissipative forces are determined exclusively by the properties of the system itself, and the corresponding forces not only affect the motion but are themselves governed by this motion since they depend on the displacements y and the velocities $\dot{y}$.

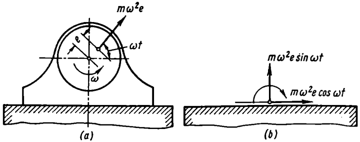


Fig. 9

Another important category of forces acting on mechanical systems are *driving forces*, i.e., forces which are given as explicit functions of time and therefore are independent of the motion but, of course, materially affect it.

An example of a driving force is provided by the vertical component of the force transmitted from an unbalanced rotor to the foundation of a machine (Fig. 9a). Let ω be the angular velocity of the rotor, m its mass and e the eccentricity. Then the magnitude of the centrifugal force is constant and equal to $m\omega^2 e$ but the direction of the force and correspondingly both components of this force as well as its moment about the middle of the bearing area vary continuously (Fig. 9b). Thus, the vertical component of the centrifugal force is

$$P = m\omega^2 e \sin \omega t;$$

such forces (and couples) may be the cause of vibrations of the elastic system on which the machine is mounted. This type of excitation is known as an *inertial* one.

In some cases driving forces may develop due to other causes, such as periodic variations of pressure in cylinders of the internal-combustion engines or periodic variations of the forces of attraction exerted by alternating electromagnets (in electric shakers).

The laws of variation of driving forces in time are widely different. We shall list some of the most important cases:

harmonic driving force (Fig. 10*a*)—machines with unbalanced rotors;

periodic driving force (Fig. 10*b*)—machines with slider-crank mechanisms;

periodic impulses of short duration (Fig. 10*c*)—vibration and shock moulding machines; in some cases the duration of each impulse is so short that they can be considered momentary (forging die hammers, impact devices, etc.);

non-periodic driving forces (Fig. 10*d*)—the engines of a rolling mill in one of the first runs of a blank, a tractor passing over an obstacle.

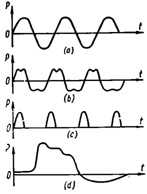


Fig. 10

It should be noted that driving forces acting on certain mechanical systems cannot be considered as specified in a definite manner, and their action should be treated as a *random process* (for example, the effect of the road on a travelling automobile).

Forces of a Mixed Nature. Complex mechanical systems may involve forces of a mixed nature, which are not resolvable into the sum of forces of the type of $P(y)$, $P(\dot{y})$, $P(t)$.

Figure 11 presents an example of a *parametric system*—a pendulum acted on by a vertical force $P = P_0 \sin \omega t$. The moment of external forces about the hinge axis is provided by the sum of the weight mg and the force P

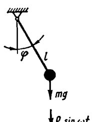


Fig. 11

$$M = -(mg + P_0 \sin \omega t)\, l \sin \varphi.$$

As is seen, this restoring moment depends not only on the displacement φ but also on the time t, and the expression $M(\varphi, t)$ cannot be represented as the sum $M_1(\varphi) + M_2(t)$, i.e., it is impossible to separate the restoring and driving components from the total moment.

There are cases where forces of the type of $P(y, \dot{y})$ are developed which are not representable as the sum $P_1(y) + P_2(\dot{y})$ (as in *self-excited vibratory systems*).

3. Methods for Setting Up Equations of Motion in the General Case

The most general form of equations of motion are *Lagrange's equations*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}=Q_i \quad (i=1,\ 2,\ \ldots,\ n), \tag{3.1}$$

where n = number of degrees of freedom of mechanical system,
i = serial number of generalized co-ordinate,
q_i = generalized co-ordinate,
$\dot{q}_i$ = generalized velocity,
Q_i = generalized force,
T = kinetic energy of system,
t = time.

For systems of several degrees of freedom, Eqs. (3.1) can be used to obtain important relations of a more special nature which provide convenient methods for investigating vibratory systems of particular types. However, such methods can also be formulated independently of Eqs. (3.1).

The following three methods are the most common.

Direct Method. In this method, we imagine the masses to be separated from the elastic framework of the system and write the differential equations of motion for each of the masses, the action of the elastic constraints being replaced by their reactions. This method is convenient in cases where the reactions of constraints (spring and damping forces) are not too difficult to express in terms of characteristic displacements and velocities, particularly for the analysis of free and forced vibrations of multimass in-line systems.

Thus, for the system shown in Fig. 12*a*, the spring forces acting on the ith mass are expressed in terms of only three displacements, x_{i-1}, x_i and x_{i+1}. Likewise, the moments of spring forces acting on any one of the solid disks of the torsional system represented in Fig. 12*b* are determined in terms of three angles of rotation, φ_{i-1}, φ_i and φ_{i+1}.

Inverse Method. The idea of this method is to separate all the masses of the system from its elastic framework and to consider the deformation of the latter under the action of the given external forces (couples) and the inertia forces (moments of inertia forces) of the separated masses.

As applied to the simplest system (Fig. 1*b*), the direct method is illustrated by Fig. 1*c* and the inverse method by Fig. 1*d*. Here R' represents the action of the mass on the spring, the projection of the force R' on the x axis being equal to the sum

$$R'=P-m\ddot{x}.$$

It may be said that the mass transmits to the spring the sum of the external force P and the inertia force $-m\ddot{x}$. Here, of course, we mean the real inertia force that is applied, however, not to the mass (Fig. 1*c*) and not to the given system (Fig. 1*b*) but to the spring which serves in this case as the elastic framework of the construction.

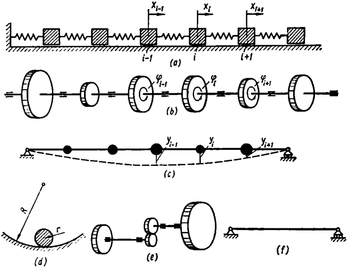


Fig. 12

Sometimes a confusion (or ambiguity) may arise in reasoning because of mixing up two different objects—the given system and its elastic framework. When speaking of inertia forces one should remember that *they are applied to the elastic framework*, i.e., to the system of constraints from which all the masses of the given system have been separated.

Energy Method. This method is based on the law of conservation of energy according to which the sum of the kinetic and potential energies is constant during vibration. It is convenient in the analysis of free vibrations of some conservative systems of one degree of freedom, such as shown in Fig. 12*d*, *e*, *f*. Figure 12*d* presents a cylindrical body of radius r resting on a cylindrical surface of radius R, Fig. 12*e* shows a system consisting of two solid disks connected by two axle shafts and a geared drive, Fig. 12*f* shows an elastic beam the mode of vibration of which is assumed to be known.

CHAPTER II

FREE VIBRATIONS

4. Linear Systems of One Degree of Freedom Without Inelastic Resistances

Standard Equation

Consider again the single-mass system shown in Fig. 1*b*. This simple system is typical of a wide class of single-degree-of-freedom systems without dissipation.

Suppose that the driving force $P(t)$ is absent but the state of equilibrium was disturbed and the system was then left alone. The motion occurring under these conditions is a free vibration. The initial disturbance of the equilibrium position is characterized by the initial conditions: $x = x_0$, $v = v_0$ at $t = 0$; the initial displacement x_0 and the initial velocity v_0 are assumed to be given (otherwise the motion will not be completely determined).

At any instant during vibration the mass is acted on by the horizontal reaction of the spring $-kx$; the value of the spring constant k is determined by the known formulas of the theory of strength of materials. The differential equation of motion (1.3) becomes

$$-kx = m\ddot{x},$$

or

$$\ddot{x} + p^2 x = 0, \tag{4.1}$$

where

$$p = \sqrt{\frac{k}{m}}. \tag{4.2}$$

Many of the vibration problems involving quite different mechanical systems reduce to an equation of the form of (4.1); we shall refer to it as the *standard equation* and consider its solution in detail.

The solution of Eq. (4.1) can be represented as

$$x = C_1 \sin pt + C_2 \cos pt$$

or in an equivalent form

$$x = a \sin (pt + \alpha). \tag{4.3}$$

The constants C_1, C_2 are determined from the initial conditions as

$$C_1 = \frac{v_0}{p}; \quad C_2 = x_0$$

so that

$$x = x_0 \cos pt + \frac{v_0}{p} \sin pt. \tag{4.4}$$

For the constants a and α, we obtain from the same conditions

$$a = \sqrt{x_0^2 + \frac{v_0^2}{p^2}}, \tag{4.4a}$$

$$\alpha = \arctan \frac{px_0}{v_0}.$$

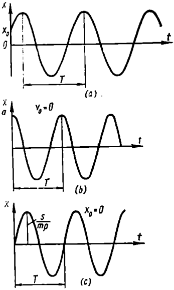


Fig. 13

From the law of motion (4.3) shown in Fig. 13*a* it is seen that the motion is a *harmonic vibration*. The process repeats itself after an interval of time T (*period of vibration*) in which the argument $pt + \alpha$ (phase) is increased by 2π

$$pt + \alpha + 2\pi = p(t + T) + \alpha.$$

Hence we find

$$T = \frac{2\pi}{p}$$

and the number of cycles per second is

$$n = \frac{1}{T} = \frac{p}{2\pi}.$$

Consequently, $p = 2\pi n$.

Hence the physical meaning of the constant p—this is the number of cycles per 2π sec. The constant p is known as the *circular* (*angular*, *cyclic*) *frequency of free vibration* or simply the *frequency of free vibration*. As is seen from formula (4.2), the frequency p depends on the parameters of the system but is independent of the nature of the initial disturbance which caused the vibratory process (of course, this is also true of the period of vibration T); for this reason the frequency of free vibration is called the *natural frequency*.

Formula (4.2) may be written in an alternate form if use is made of the concept of a static displacement

$$f_{st} = \frac{mg}{k}.$$

By f_{st} is meant the displacement of the end of the spring which would be produced by the weight mg acting along the axis of the spring. The formula for the natural frequency then becomes

$$p = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{f_{st}}}$$

giving the expression for the number of cycles per second as

$$n = \frac{1}{2\pi}\sqrt{\frac{g}{f_{st}}} \cong \frac{5}{\sqrt{f_{st}}}$$

(f_{st} in cm).

Except the frequency, all the characteristics of the process of free vibration are greatly influenced by the initial conditions. Thus, if the vibrations are set up when the mass is pulled out a distance a from the equilibrium position and then released, then $x_0 = a$, $v_0 = 0$ and, according to Eq. (4.4), the motion follows a cosine law

$$x = a \cos pt,$$

as is illustrated in Fig. 13*b*. If the vibrations result from a momentary impact on the mass, the initial conditions are

$$x_0 = 0, \quad v_0 = \frac{S}{m},$$

where S is the impact impulse (the second initial condition is formulated on the basis of the theorem of impulse and momentum). According to Eq. (4.4), the motion of the mass follows a sine law

$$x = \frac{S}{mp} \sin pt$$

with $x_{\max} = \frac{S}{mp}$ (Fig. 13*c*).

When x_0 and v_0 are not zero simultaneously, the motion is governed by law (4.4).

It should be noted that the solution obtained has an obvious shortcoming; since no account has been taken of unavoidable inelastic resistances (friction), the resulting vibrations involve no damping. This shortcoming will be given consideration below and corrected; however, it may be stated that the effect of these resistances on the natural frequency is generally *very slight*. Formula (4.2) is therefore in fair agreement with experimental data.

The law of motion can be illustrated not only by curves in the time-displacement co-ordinate system. In some cases the *phase* *plane*, i.e., the displacement-velocity co-ordinate system may conveniently be used to describe and study the motion.

At any given instant the state of a system is characterized by the displacement x and the velocity v; to this state corresponds a *generic point* of co-ordinates x, v in the phase plane. The generic point will move with time on the phase plane describing a *phase trajectory*.

In the above case of harmonic vibration we have

$$\left.\begin{aligned} x &= a\sin(pt+\alpha), \\ v &= ap\cos(pt+\alpha). \end{aligned}\right\} \tag{4.5}$$

This set of equations may be regarded as a phase trajectory prescribed in parametric form (with the time t as a parameter). To obtain the equation of the phase trajectory in explicit form, it is necessary to eliminate the time t from system (4.5); we obtain

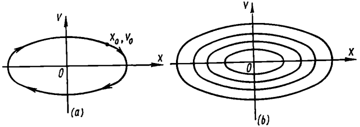


Fig. 14

$$\frac{x^2}{a^2} + \frac{v^2}{a^2p^2} = 1,$$

i.e., the equation of an ellipse (Fig. 14*a*).

To the initial conditions $x = x_0$, $v = v_0$ corresponds the initial generic point of the phase trajectory from which the motion is started. The periodicity of the process expresses itself in the fact that the generic point will run round one and the same elliptic orbit.

When the initial conditions are changed, the phase trajectory is found to be a different ellipse; the set of all possible states of the system is described by a family of ellipses embedded into one another (Fig. 14*b*). The set of phase trajectories forms the *phase diagram* of the system. It may be said that the parameters of the system determine the phase diagram and the initial conditions fix a particular trajectory.

The origin of co-ordinates corresponds to the state of equilibrium. If $x_0 = 0$, $v_0 = 0$, the generic point is always at the origin and describes no trajectory; the points of equilibrium are called *singular points*. If a singular point is surrounded by a system of closed trajectories (as in the case under consideration), it is called a singular point of the *centre* type.

The above system is a typical example, since free vibrations of any linear systems of one degree of freedom (without damping) are described by exactly the same differential equation.

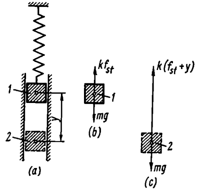


Fig. 15

Note that the natural frequency of the system is independent of the orientation of the spring axis. In particular, consider the case when the spring is suspended vertically (Fig. 15*a*).

Let point *1* define the equilibrium position of the mass (Fig. 15*b*). In this position the mass is acted on by its weight mg and the reaction of the spring kf_{st} (f_{st} is the static extension of the spring corresponding to the weight of the mass mg). The condition of equilibrium of the mass at rest is

$$mg - kf_{st} = 0,$$

i. e.,

$$kf_{st} = mg. \tag{4.6}$$

If the equilibrium is disturbed and the mass is in motion, at any given instant the mass is acted on by its weight mg and the reaction of the spring $k(f_{st} + y)$, where y is the deflection of the mass from its equilibrium position (Fig. 15*c*).

The differential equation of motion

$$-k\,(f_{st}+y)+mg=m\ddot{y}$$

with condition (4.6) taken into account, assumes again the standard form (4.1) if x is substituted for y.

The coincidence of the differential equations means that the whole vibratory process occurs in the same way as in the case of the horizontal position of the spring axis (of course, provided the initial conditions are similar); the centre of oscillation is located at the equilibrium position (point *1*).

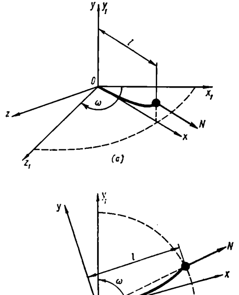


Fig. 16

The same results are obtained if the mass is suspended from any other elastic structure (provided it executes vibrations along a straight line and the structure can be considered weightless). The features of each particular case are accounted for only by a spring constant k, which depends on the elastic system at hand. The values of k for some simple cases are given in Table 1.

Consider cases 10 and 11. These diagrams correspond to the conditions involved in the rotation of a cantilever beam about an axis through the centroid of the root section when the *centrifugal force* $m\omega^2 l$ is developed (ω is the angular velocity). If the axis of rotation coincides with the y axis (Fig. 16*a*), the direction of the centrifugal force remains constant and we have to use diagram 10. If, however, the axis of rotation is perpendicular to the xy plane, the centrifugal force must pass through the centroid of the root section (Fig. 16*b*) and the spring constant k is determined according to diagram 11.

Let us now turn to systems containing rigid bodies which execute *angular vibrations* about a fixed axis (Fig. 17). The first system

Table 1

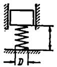
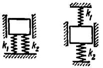
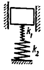
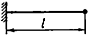
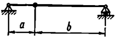
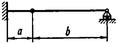
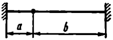
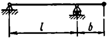
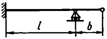
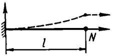
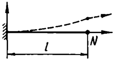
Item No.	Schematic Diagram	Spring Constant, k
1		$\frac{Gd^4}{8nD^3}$ (d = diameter of coil section, D = diameter of spring, G = shear modulus, n = number of coils)
2		$k_1 + k_2$
3		$\frac{k_1 k_2}{k_1 + k_2}$

Table 1 (continued)

Item No.	Schematic Diagram	Spring Constant, k (EI = flexural rigidity)	
4		$\frac{3EI}{l^3}$	
5		$\frac{3EI\,(a+b)}{a^2b^2}$	
6		$\frac{12EI\,(a+b)^3}{a^3b^2\,(3a+4b)}$	
7		$\frac{3EI\,(a+b)^3}{a^3b^3}$	
8		$\frac{3EI}{(b+l)\,b^2}$	
9		$\frac{12EI}{(4b+3l)\,b^2}$	
10		$\frac{\alpha^3EI\cosh\alpha l}{\alpha l\cosh\alpha l-\sinh\alpha l}$	$\alpha=\sqrt{\frac{N}{EI}}$
11		$\frac{\alpha^2EI\sinh\alpha l}{l\,(\alpha l\cosh\alpha l-\sinh\alpha l)}$	

(Fig. 17*a*) represents a disk mouted on an elastic rod, and the second one (Fig. 17*b*), a two-dimensional elastically suspended rigid lever having one point fixed. The third system (Fig. 17*c*) consists of a heavy flywheel connected to the shaft by a flexible spiral spring; this system is basically the same as the first one.

In all the foregoing cases we proceed from the differential equation of motion of a body rotating about a fixed axis

$$M = J\ddot{\varphi}, \tag{4.7}$$

where M = moment of forces applied to body about axis of rotation,

J = moment of inertia of body about the same axis.

In the first of the above systems the moment M (restoring moment) is produced by the elastic forces of the rod and is equal to $-k\varphi$,

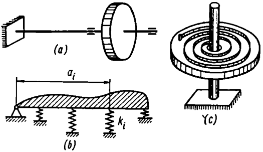


Fig. 17

where $k = GI_p/l$ is the *torsional stiffness* (GI_p is the torsional rigidity of the material, l is the length of the twisted rod). Consequently, according to Eq. (4.7)

$$-k\varphi = J\ddot{\varphi},$$

i. e.,

$$\ddot{\varphi} + p^2\varphi = 0, \tag{4.8}$$

where

$$p = \sqrt{\frac{k}{J}}. \tag{4.9}$$

The differential equation (4.8) is of the standard form (4.1); it follows immediately that formula (4.9) determines the *natural frequency of vibration*.

If the angles of rotation φ of the beam shown in Fig. 17*b* are measured from the equilibrium position, the small additional extension of each of the springs during vibration is written as $a_i\varphi$ (a_i is the distance from the point of attachment of the spring to the hinge support), and the additional reaction of the spring is written as $-k_i a_i \varphi$ (k_i is the stiffness of the ith spring). Accordingly the moment produced by one spring is $-k_i a_i^2 \varphi$ and the total moment is

$$M = -\Sigma k_i a_i^2 \varphi = -\varphi \Sigma k_i a_i^2 = -k\varphi \qquad (4.10)$$

with the sum

$$k = \Sigma k_i a_i^2$$

as the combined spring stiffness of the system.

Substituting expression (4.10) into the differential equation (4.7), we arrive again at the standard form of writing (4.1).

Consider now the two-mass system represented in Fig. 2*p*, the features of which have already been mentioned. The onset of the vibratory process can be imagined, for example, as follows. Let the disks be acted on by two equal and opposite torques which are suddenly released at a certain instant (taken as the origin of time). At a time $t > 0$ the angles of rotation of the disks are φ_1 and φ_2 so that the relative angle of rotation is $\varphi_2 - \varphi_1$. The moment of the elastic forces of the shaft is equal to $k(\varphi_2 - \varphi_1)$ and acts on either of the disks as shown in Fig. 18. Denoting by J_1 and J_2 the mass moments of inertia of the disks about the shaft axis, we obtain the equations of motion

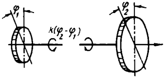


Fig. 18

$$k(\varphi_2 - \varphi_1) = J_1 \ddot{\varphi}_1,$$

$$-k(\varphi_2 - \varphi_1) = J_2 \ddot{\varphi}_2.$$

The minus sign on the left-hand side of the second equation is introduced because the elastic moment is directed clockwise (negative moment) for the second disk. Dividing the first equation by J_1, the second equation by J_2 and subtracting the first equation from the second, we obtain

$$-\left(\frac{k}{J_2} + \frac{k}{J_1}\right)(\varphi_2 - \varphi_1) = \ddot{\varphi}_2 - \ddot{\varphi}_1$$

Introduce in the equation the relative angle of rotation of the disks

$$\varphi = \varphi_2 - \varphi_1.$$

Then the equation takes the earlier form [see Eq. (4.8)], with the *natural frequency*

$$p=\sqrt{\frac{k(J_1+J_2)}{J_1J_2}}\,. \tag{4.11}$$

The results will remain unchanged if, in addition to elastic vibrations, the whole system rotates as a rigid body.

When $J_1 \to \infty$, we arrive again at the formula (4.9) derived for the shaft with the left end fixed. For finite values of J_1 and J_2 a certain intermediate section, called a *node of vibration*, does not participate in vibration. To locate the node of vibration we note that no external moments act during vibration; hence, at any instant the sum of the moments of the inertia forces of both disks about the shaft axis must be zero, i.e.,

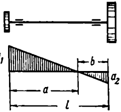


Fig. 19

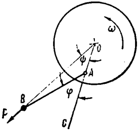


Fig. 20

$$-J_1\ddot{\varphi}_1-J_2\ddot{\varphi}_2=0.$$

Denoting by a_1 and a_2 the amplitudes of angular displacements, we obtain the accelerations

$$\ddot{\varphi}_1=-a_1p^2\sin(pt+\alpha),$$

$$\ddot{\varphi}_2=-a_2p^2\sin(pt+\alpha),$$

therefore

$$J_1a_1+J_2a_2=0,$$

whence

$$\frac{a_1}{a_2}=-\frac{J_2}{J_1}\,,$$

i.e., the amplitudes of vibration of the disks are in the inverse ratio of their moments of inertia; the minus sign indicates that the disks rotate in opposite directions during the vibration. This result is shown in Fig. 19 from which it is seen that the distances from the node to the ends of the shaft are

$$a=\frac{J_2}{J_1+J_2}\,l,\quad b=\frac{J_1}{J_1+J_3}\,l. \tag{4.12}$$

Example 1. Determine the natural frequency of torsional vibration of a two-mass system (Fig. 19) when the following data are given: diameters of disks are $d_1 = 30$ cm and $d_2 = 20$ cm; thicknesses of disks $b_1 = 2$ cm and $b_2 = 1.5$ cm; diameter of shaft $d_0 = 1$ cm; length of shaft $l = 80$ cm.

The material of the disks and shaft is steel ($\gamma = 0.0078$ kgf/cm³, $G = 0.8 \times 10^6$ kgf/cm²).

The mass moments of inertia of the disks are

$$J_1=b_1\frac{\pi d_1^4}{32}\,\frac{\gamma}{g}=2\,\frac{3.14\times 30^4\times 0.0078}{32\times 981}=1.262\ \text{kgf-cm-sec}^2,$$

$$J_2=b_2\frac{\pi d_2^4}{32}\,\frac{\gamma}{g}=1.5\,\frac{3.14\times 20^4\times 0.0078}{32\times 981}=0.187\ \text{kgf-cm-sec}^2.$$

The polar moment of inertia of the cross section of the shaft is

$$I_p=\frac{\pi d_0^4}{32}=\frac{3.14\times 1^4}{32}=0.0981\ \text{cm}^4.$$

The torsional stiffness of the shaft is

$$k=\frac{GI_p}{l}=\frac{0.8\times 10^6\times 0.0981}{80}=981\ \text{kgf-cm}.$$

The natural frequency, by formula (4.11), is

$$p=\sqrt{\frac{981\,(1.262+0.187)}{1.262\times 0.187}}=78\ \text{sec}^{-1}.$$

The node is located near the larger disk; by formula (4.12)

$$a=\frac{0.187}{0.187+1.262}\,80=10.3\ \text{cm},\quad b=\frac{1.262}{0.187+1.262}\,80=69.7\ \text{cm}.$$

The shaft may be assumed to be built into the larger disk; by formula (4.9) we find then the natural frequency with an error of about 7 per cent.

Of great practical significance is the problem of *oscillation of a pendulum in the centrifugal field.*

Consider a disk rotating with a constant angular velocity ω (Fig. 20). To a point A of the disk is attached a weight B by means of a weightless rod AB. Let the pendulum AB be displaced from its equilibrium position through a small angle φ. Determine the frequency of oscillation of the weight about the equilibrium position AC. The weight B participates in two motions—in base motion with the disk and in relative motion about the centre of oscillation A. Denote by ψ the angle that the direction OB makes with the direction OA, by R the distance from the centre of the disk O to the centre of oscillation A, by l the length of the pendulum AB, by L the distance OB, by m the mass of the weight B.

Then

$$L=\frac{R+l\cos\varphi}{\cos\psi}.$$

In analysing the relative motion of the pendulum AB it is necessary to consider the base and Coriolis inertia forces. The base inertia force is directed along the straight line OB and is equal to

$$P=m\omega^2L=m\omega^2\frac{R+l\cos\varphi}{\cos\psi}. \tag{4.13}$$

The Coriolis inertia force directed along the straight line AB need not be determined as we propose to set up the moment equation with respect to point A. The differential equation of relative motion of the pendulum is

$$-PR\sin\psi=ml^2\ddot{\varphi}.$$

Substituting expression (4.13), we obtain

$$m\omega^2R\,(R+l\cos\varphi)\tan\psi+ml^2\ddot{\varphi}=0. \tag{4.14}$$

Since

$$\tan\psi=\frac{x}{R+l\cos\varphi},$$

where x is the distance of point B from line OA, Eq. (4.14) may be written as

$$\omega^2Rx+l^2\ddot{\varphi}=0. \tag{4.15}$$

If the oscillations are small enough, it may be taken that $\varphi\cong x/l$, and Eq. (4.15) will be

$$\ddot{x}+\frac{\omega^2R}{l}x=0.$$

Here, as in the foregoing cases, it is sufficient to carry the computations to the differential equation of the standard form, whereupon from the coefficient of the co-ordinate (co-ordinate x in the present case) we determine immediately the natural frequency

$$p=\omega\sqrt{\frac{R}{l}}. \tag{4.16}$$

Note that the natural frequency is proportional to the angular velocity of the disk.

The above result may be used to determine the natural frequency of oscillation of a *bifilar-type pendulum* (Fig. 21*a*). The suspension is accomplished by means of two rollers of diameter d_2 inserted in somewhat larger holes of diameter d_1 which are provided in the pendulum and the rotating disk. With this method of suspension the relative motion of the pendulum (with respect to the rotating disk) is a translation and all its points describe circular arcs of the same radius. This radius is equal to the difference between the diameters of the hole and roller, i.e., $l = d_1 - d_2$. Also, the design dimension R in this case is $R_1 - l$, where R_1 is the distance from the centre of the rotating disk to the centre of gravity of the pendulum. Thus, the natural frequency of oscillation of the pendulum is given by the formula

$$p = \omega \sqrt{\frac{R_1}{d_1 - d_2} - 1}. \qquad (4.17)$$

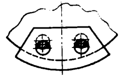


Fig. 21

It is well to recall here that in deriving formula (4.16) it was assumed that the ratio of the displacement x to the length l of the pendulum was small. In the present case the design length of the pendulum is itself small; this imposes particularly severe restrictions on the magnitude of the amplitudes of oscillation of the pendulum, and if the ratio x/l cannot be considered small in comparison with unity, the linear theory should not be used at all.

The identity of the differential equations for various kinds of systems is not coincidental. The fact is that for small vibrations of mechanical systems of one degree of freedom the kinetic and potential energies are defined by the expressions

$$T = \frac{1}{2} a\dot{q}^2, \quad U = \frac{1}{2} kq^2,$$

where a is the inertia coefficient, k the elastic coefficient (the total spring stiffness), q the generalized co-ordinate. Accordingly, from Lagrange's equation is obtained

$$a\ddot{q} + kq = 0$$

which equation is of the same type as the standard equation (4.1) $\left(\text{with } p^2 = \frac{k}{a}\right)$.

Energy Method. Rayleigh's and Grammel's Formulas

Apart from the simplest system of Fig. 1, in all the foregoing cases the computations were not carried beyond the derivation of the differential equation of motion of the system. After reduction of this equation to the standard form, it was possible, with the expression for the coefficient of the co-ordinate, to write down immediately the formula for the natural frequency of vibration

In some cases it is more expeditious to use an alternate method of determining the natural frequency; this method requires no derivation of the differential equation of motion and is based on energy considerations. Assuming that the motion defined by the co-ordinate is harmonic and is described by the expression

$$y = a \sin (pt + \alpha),$$

it is possible to determine the velocity

$$\dot{y} = ap \cos (pt + \alpha).$$

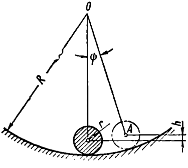


Fig. 22

The maximum potential energy $U_{\max}$ of the system is attained at the time of maximum deflection of the system from its equilibrium position and is determined by this deflection, equal to a. Accordingly the maximum kinetic energy $T_{\max}$ is determined by the maxium velocity $v_{\max} = ap$ which is reached when the system passes the equilibrium position.

If the potential energy is reckoned from the equilibrium position, the following equality must be fulfilled

$$U_{\max} = T_{\max}$$

from which the natural frequency p may be determined.

Considor two cases where the natural frequency may be found most conveniently by the energy method. In the first case the restoring force is produced by an elastic spring (see Fig. 2m), in the second by the gravity force of a cylinder (Fig. 22).

Consider first the vibration of a solid cylinder which may roll freely and without sliding on a horizontal plane (see Fig. 2m). Let r be the radius of tho cylinder, m its mass, ρ the radius of gyration and k the spring constant.

The maximum potential energy is

$$U_{\max} = \frac{ka^2}{2}. \tag{4.18}$$

The kinetic energy should be calculated with due allowance for rotation

$$T_{\max} = \frac{mv_{\max}^2}{2} + \frac{m\rho^2\omega_{\max}^2}{2}.$$

Replacing

$$\omega_{\max} = \frac{v_{\max}}{r}, \quad v_{\max} = ap,$$

we obtain

$$T_{\max}=\frac{ma^2p^2}{2}+\frac{ma^2p^2\rho^2}{2r^2}=\frac{ma^2p^2}{2}\left(1+\frac{\rho^2}{r^2}\right). \qquad (4.19)$$

Equating expressions (4.18) and (4.19), we find the natural frequency of the vibration

$$p=\sqrt{\frac{k}{m\left(1+\frac{\rho^2}{r^2}\right)}}. \qquad (4.20)$$

From formula (4.20) it is seen that the natural frequency is reduced due to the rotary inertia of the cylinder.

If the cylinder is homogeneous, then $\rho^2=0.5r^2$ and the natural frequency is 18 per cent less than in the case of the same mass having only a motion of translation, but no rotation.

We now turn to Fig. 22 representing a solid cylinder which may roll freely along a cylindrical surface of radius R; the radius of the cylinder is r. Let ψ be the maximum angle of deflection during vibration of the segment OA drawn from the centre of curvature O to the centre A of the section of the cylinder.

The maximum height h to which the centre of gravity of the cylinder rises is then

$$h=(R-r)(1-\cos\psi).$$

For small angles ψ it is permissible to put

$$\cos\psi\cong1-\frac{\psi^2}{2},$$

i.e.,

$$h=\frac{R-r}{2}\psi^2;$$

the maximum potential energy is

$$U_{\max}=\frac{G(R-r)}{2}\psi^2, \qquad (4.21)$$

where G is the weight of the cylinder.

To determine the maximum kinetic energy we shall first find the maximum velocity of the centre A of the cylinder

$$v_{\max}=\psi p(R-r)$$

and the corresponding maximum angular velocity of the cylinder

$$\omega_{\max}=\frac{v_{\max}}{r}=\frac{\psi p(R-r)}{r}.$$

Consequently,

$$T_{\max}=\frac{G}{2g}[\psi p(R-r)]^2+\frac{G\rho^2}{2g}\left[\frac{\psi p(R-r)}{r}\right]^2, \qquad (4.22)$$

where ρ is the radius of gyration of the cylinder about its axis.

Equating expressions (4.21) and (4.22), we obtain the natural frequency

$$p=\sqrt{\frac{g}{(R-r)\left(1+\frac{\rho^2}{r^2}\right)}}\ .$$

If the cylinder is homogeneous, then $\rho^2=0.5r^2$ and accordingly

$$p=\sqrt{\frac{2g}{3(R-r)}}\ .$$

By varying the difference of the radii $R-r$ the frequency may be changed from the minimum value

$$p=\sqrt{\frac{2g}{3R}}$$

(for a vanishingly small radius of the cylinder) to infinity.

As already noted, specifying a particular *mode of vibration* of a system with distributed mass and elasticity, we thereby assign one degree of freedom to it. The natural frequency of vibration of this schematized system may also be conveniently determined by the *energy method* (called *Rayleigh's method* in this case).

Of course, the results will then depend on the choice of mode of vibration, and the solution will no longer possess uniqueness, as was the case in the preceding two examples.

Consider the most general form of Rayleigh's method as applied to the problem of *transverse vibration of a beam*. Assume that the displacements of the points of the beam axis are described by the law

$$y(x,\ t)=f(x)\sin(pt+\alpha), \tag{4.23}$$

i.e., the vibrations of all the points of the axis have the same frequency and are in phase (all the points pass through the equilibrium position simultaneously, reach the maximum deflections at the same instant and so on).

The function $f(s)$ represents the mode of vibration, i.e., it defines the configuration of the elastic curve when the deflections reach their maximum values

$$y_{\max}(x)=f(x).$$

According to law (4.23), the velocities of the points of the beam axis are given by the relation

$$v(x,\ t)=\frac{\partial y}{\partial t}=pf(x)\cos(pt+\alpha)$$

and the maximum velocities (when the system passes through the equilibrium position) are

$$v_{\max}(x)=pf(x).$$

The maximum potential energy is

$$U_{\max} = \frac{1}{2} \int\limits_0^l EI\,(y''_{\max})^2\,dx = \frac{1}{2} \int\limits_0^l EI\,(f'')^2\,dx. \tag{4.24}$$

The maximum kinetic energy is

$$T_{\max} = \frac{1}{2} \int\limits_0^l m v_{\max}^2\,dx = \frac{p^2}{2} \int\limits_0^l m f^2\,dx, \tag{4.25}$$

where $m = m\,(x)$ is the intensity of the distributed mass of the beam.

Equating expressions (4.24) and (4.25), we obtain Rayleigh's fundamental formula for the case of transverse vibration

$$p^2 = \frac{\int\limits_0^l EI\,(f'')^2\,dx}{\int\limits_0^l m f^2\,dx}. \tag{4.26}$$

If, in addition to the distributed mass, the beam carries concentrated masses m_i at sections x_i, the kinetic energy expression will then include terms of the type $\frac{1}{2} m_i f_i^2 p^2$, where f_i is the value of the function $f\,(x)$ at the point with abscissa x_i; formula (4.26) then becomes

$$p^2 = \frac{\int\limits_0^l EI\,(f'')^2\,dx}{\int\limits_0^l m f^2\,dx + \Sigma m_i f_i^2}. \tag{4.27}$$

The foregoing variants of Rayleigh's formula give exact values if the true mode of vibration $f\,(x)$ is substituted into the formula. However, this mode is not known in advance and hence, when using Rayleigh's formula in practice, one should *assign* the mode of vibration; this introduces some inaccuracy in the results.

Not only did Rayleigh propose the energy method but he also proved an important theorem: the value of the natural frequency obtained by the energy method is always higher than or equal to the true one (see below, pp. 139-141).

The scale of the function $f\,(x)$ is unimportant; the multiplication of $f\,(x)$ by any number will not change the result, as is seen from the structure of formula (4.26). Care must be taken only to reflect the expected mode of vibration as best as possible and in any case to ensure the fulfilment of the boundary conditions corresponding to the prescribed end conditions for the beam (geometric boundary conditions).

Example 2. Determine the natural frequency of vibration of a simply supported beam carrying three equal masses m (Fig. 23).

We assign the mode of vibration as

$$f = a \sin \frac{\pi x}{l};$$

this expression satisfies all the boundary conditions ($f = f'' = 0$ for $x = 0$ and $x = l$) and defines a symmetric curve. The quantity a remains undetermined [a will appear as a factor in both the numerator and denominator of expression (4.27) and will cancel out without affecting the results of calculations].

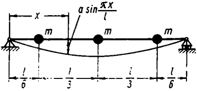


Fig. 23

We find

$$f'' = -a \frac{\pi^2}{l^2} \sin \frac{\pi x}{l},$$

$$(f'')^2 = a^2 \frac{\pi^4}{l^4} \sin^2 \frac{\pi x}{l}.$$

The numerator in formula (4.27) is

$$\int_0^l EI\,(f'')^2\,dx = \frac{a^2 \pi^4 EI}{2l^3}.$$

To find the denominator we calculate the deflections under the masses

$$f_1 = f_3 = a \sin \frac{\pi}{l} \frac{l}{6} = \frac{a}{2},$$

$$f_2 = a \sin \frac{\pi}{l} \frac{l}{2} = a.$$

The denominator in formula (4.27) consists of only the second term

$$\sum m_i f_i^2 = m \left(\frac{a}{2}\right)^2 + ma^2 + m \left(\frac{a}{2}\right)^2 = \frac{3}{2} ma^2.$$

The square of the natural frequency is, by formula (4.27),

$$p^2 = \frac{\pi^4 EI}{3ml^3}$$

giving the natural frequency as

$$p = \frac{5.696}{l} \sqrt{\frac{EI}{ml}}.$$

We note that this result differs only slightly from the exact value

$$p = \frac{5.692}{l} \sqrt{\frac{EI}{ml}}.$$

One popular procedure of using Rayleigh's formula is to assign a *fictitious load* $q(x)$ rather than a function $f(x)$; the static-deflection curve due to this load $q(x)$ is substituted in Rayleigh's formula. The advantage of this procedure lies in the fact that the boundary conditions are automatically satisfied and, besides, the calculation of the maximum potential energy by formula (4.24) can be replaced by a simpler calculation of the work done by the static load

$$W=\frac{1}{2}\int_0^l qf\,dx \tag{4.28}$$

as the quantities W and $U_{\max}$ are equal. Formula (4.27) then becomes

$$p^2=\frac{\int_0^l qf\,dx}{\int_0^l mf^2\,dx+\Sigma m_i f_i^2}. \tag{4.29}$$

Of course, the fictitious load may also include concentrated forces P_i; formula (4.29) is then written as

$$p^2=\frac{\int_0^l qf\,dx+\Sigma P_i f_i}{\int_0^l mf^2\,dx+\Sigma m_i f_i^2}. \tag{4.30}$$

It should be noted that the quantities m_i and P_i are not related; the former represent the actually existing concentrated masses while the latter represent the forces constituting the "imaginary" load; similarly, the functions $m(x)$ and $q(x)$ are not related.

The last variants of Rayleigh's formula are basically equivalent to formula (4.27) but, involving no operation of differentiation, they generally provide better accuracy.

The fictitious load defining the mode of vibration is most commonly assumed according to either of the following variations:

a concentrated force applied at some characteristic point of the beam axis;

actually acting forces of weight of the beam.

Consider the *first variation.* Let some characteristic point be chosen and f_0 be the deflection of this point ("point of reduction"); the maximum potential energy due to bending can then be expressed by

$$U_{\max}=\frac{kf_0^2}{2},$$

where k is the spring constant (corresponding to the chosen point of reduction).

Consequently, in place of formula (4.29) we can write

$$p^2=\frac{kf_0^2}{\int_0^l mf^2\,dx}.$$

Denote

$$m_0 = \frac{\int\limits_0^l mf^2\,dx}{f_0^2}\,; \tag{4.31}$$

we then obtain the familiar form of writing

$$p^2 = \frac{k}{m_0} \tag{4.32}$$

as if the mass m_0 were concentrated at the point of reduction; the quantity m_0 may be called the *reduced mass*.

If, in addition to the distributed mass, there are also concentrated masses, the expression for the reduced mass becomes

$$m_0 = \frac{\int\limits_0^l mf^2\,dx + \Sigma m_i f_i^2}{f_0^2}\,. \tag{4.33}$$

After determining the reduced mass by formulas (4.31) or (4.33), we find the natural frequency of vibration by formula (4.32).

Of course, $f(x)$ can no longer be chosen arbitrarily in expressions (4.31) and (4.33); $f(x)$ is the function defining the deflections of the axis due to the concentrated force (the scale of this function is unimportant as is seen from the structure of the formula for m_0).

The concept of the reduced mass eliminates to some extent the arbitrary choice mentioned above (true, the choice of point of reduction remains arbitrary). However, in distinction to formulas (4.27) or (4.30), formula (4.32) can never yield the exact solution as the deflection curve due to the concentrated force does not coincide with the true deflection form during vibration.

Example 3. Determine the reduced mass for the beam represented in Fig. 23 and find the natural frequency of vibration. Choose the point of reduction at the middle of the beam.

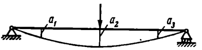


Fig. 24

Formula (4.33) becomes in this case

$$m_0 = \frac{\Sigma m_i f_i^2}{f_0^2} = m\frac{\Sigma f_i^2}{f_0^2}\,. \tag{4.34}$$

Referring to Fig. 24 we find

$$f_1 = f_3 = \frac{39}{81}f_0, \quad f_2 = f_0$$

(the magnitude of the force and accordingly the scale of f_0 are unimportant).

Using formula (4.34) we obtain

$$m_0 = m\left(\frac{39}{81}\right)^2 \times 2 + m = 1.4636\ m.$$

The spring constant for the system (see Item 5 in Table 1) is

$$k = \frac{48EI}{l^3}\,.$$

The square of the natural frequency is, by formula (4.32),

$$p^2 = \frac{48EI}{1.4636ml^3} = 32.79\,\frac{EI}{ml^3}\,.$$

Consequently,

$$p = \frac{5.726}{l}\sqrt{\frac{EI}{ml}}\,.$$

Better results are obtained if use is made of the *second variation* assuming the fictitious load in formula (4.30) to be the forces of weight of the beam, $q(x) = m(x)\,g$, and the forces P_i to be the actual weights $m_i g$ of the concentrated masses. This variation corresponds to the assumption that the mode of vibration coincides with the static-deflection curve due to the weight of the beam itself and the masses attached to it. Formula (4.30) is then written as

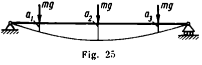


Fig. 25

$$p^2 = g\,\frac{\int\limits_0^l mf\,dx + \Sigma m_i f_i}{\int\limits_0^l mf^2\,dx + \Sigma m_i f_i^2}\,. \tag{4.35}$$

When using this formula, the function $f(x)$ is no longer chosen; it represents the well defined static-deflection curve, therefore the last variant of Rayleigh's formula possesses complete definiteness but, as with formula (4.32), it can never give the exact value of the frequency p.

Example 4. Determine the natural frequency for the beam represented in Fig. 23 by Rayleigh's formula (4.35).

In this case Rayleigh's formula takes the form

$$p^2 = g\,\frac{\Sigma m_i f_i}{\Sigma m_i f_i^2} = g\,\frac{\Sigma f_i}{\Sigma f_i^2}\,.$$

After finding the static deflections under the masses (Fig. 25)

$$f_1 = f_3 = \frac{mgl^3}{48EI}\,, \quad f_2 = \frac{53mgl^3}{1{,}296EI}\,,$$

we calculate the numerator and denominator of the expression for p^2

$$\Sigma f_i = \frac{mgl^3}{EI}\left(\frac{1}{48} + \frac{53}{1{,}296} + \frac{1}{48}\right) = \frac{107mgl^3}{1{,}296EI}\,,$$

$$\Sigma f_i^2 = \left(\frac{mgl^3}{EI}\right)^2\left[\left(\frac{1}{48}\right)^2 + \left(\frac{53}{1{,}296}\right)^2 + \left(\frac{1}{48}\right)^2\right] = \frac{4{,}267m^2g^2l^6}{1{,}296^2E^2I^2}\,.$$

The square of the natural frequency is

$$p^2 = \frac{107 \times 1{,}296}{4{,}267} \frac{EI}{ml^3},$$

whence

$$p = \frac{5.701}{l} \sqrt{\frac{EI}{ml}}.$$

A useful modification of the energy method was proposed by Grammel in the following form. Let $f(x)$ be the assigned mode of free vibration of the bar; then the intensity of the corresponding maximum inertia forces is given by the expression mp^2f, where $m = m(x)$ is the intensity of the distributed mass as before.

We write the expression for the maximum potential energy due to bending in terms of the bending moments produced by the maximum inertia forces

$$U_{\max} = \frac{1}{2} \int_0^l \frac{M_b^2\, dx}{EI}. \tag{4.36}$$

Here $M_b = M_b(x)$ are the bending moments produced by the load mp^2f. Denote by $\overline{M}_b$ the bending moment produced by an assumed load mf (i.e., by a load p^2 times smaller than the inertia forces); then $M_b = p^2\overline{M}_b$ and expression (4.36) may be written as

$$U_{\max} = \frac{p^4}{2} \int_0^l \frac{\overline{M}_b^2\, dx}{EI}. \tag{4.37}$$

The maximum kinetic energy is defined, as before, by the expression

$$T_{\max} = \frac{p^2}{2} \int_0^l mf^2\, dx. \tag{4.38}$$

Equating expressions (4.37) and (4.38), we arrive at *Grammel's formula*

$$p^2 = \frac{\int_0^l mf^2\, dx}{\int_0^l \frac{\overline{M}_b^2\, dx}{EI}}. \tag{4.39}$$

In using this formula first assign an appropriate function $f(x)$ so as to approximate the expected mode of free vibration as closely as possible and to satisfy the boundary conditions. Next, determine the assumed load mf by multiplying the function $f(x)$ by the function $m(x)$ known from the conditions of the problem, and then find the bending moments $\overline{M}_b$ produced by the load assumed above using the well-known methods of strength of materials. It now remains to calculate the expressions appearing in the numerator and denominator of formula (4.39).

Grammel's formula involves a somewhat greater amount of numerical work than Rayleigh's formula but gives a closer approximation with the same choice of mode of vibration $f(x)$.

This can be illustrated by a simple example of free vibration of a cantilever beam of uniform section. Let the left end of the beam be fixed (place the origin of co-ordinates there) and the right end free. Assume the mode of vibration to be

$$f(x) = ax^2,$$

where x = co-ordinate of section,

a = a constant (its value is immaterial as it cancels in final expressions).

This function satisfies the geometric boundary conditions and can be used as the basis for calculations by either Rayleigh's or Grammel's formula.

To use *Rayleigh's formula* we first find

$$\left.\begin{aligned} \int_0^l EI\,(f'')^2\,dx &= 4a^2 lEI, \\ \int_0^l mf^2\,dx &= \frac{ma^2l^5}{5}. \end{aligned}\right\} \tag{4.40}$$

Now determine the square of the natural frequency of vibration by formula (4.26)

$$p^2 = \frac{20EI}{ml^4}.$$

Note that this result differs markedly from the exact value

$$p^2 = \frac{12.36EI}{ml^4}.$$

To calculate the natural frequency by Grammel's formula we choose the assumed load as max^2 and find the corresponding bending moments due to this load

$$\overline{M}_b = \frac{ma}{12}\,(x^4 - 4xl^3 + 3l^4).$$

We now determine the denominator of expression (4.39)

$$\int_0^l \frac{\overline{M}_b^2\,dx}{EI} = \frac{m^2a^2l^9}{62.31EI}. \tag{4.41}$$

The numerator of expression (4.39) has already been found as (4.40). Dividing the second of formulas (4.40) by (4.41), we find

$$p^2 = \frac{12.46EI}{ml^4}$$

which value is *considerably* closer to the exact value than the result obtained by Rayleigh's formula.

The better accuracy of Grammel's formula is due to the fact that it involves no operation of differentiation, which operation always accentuates the error of approximate calculations.

The energy method enables one not only to obtain immediately satisfactory results but also to make them as accurate as desired by means of *successive approximations*. Consider the case of flexural vibration and assume that there is a beam carrying n masses, m_1, m_2, . . ., m_n. The law governing the vibration of any point is

$$y_i = a_i \sin(pt + \alpha).$$

The corresponding inertia forces are

$$-m_i\ddot{y}_i = m_i a_i p^2 \sin(pt + \alpha).$$

In the extreme deflected position these forces take the values

$$F_i = m_i a_i p^2.$$

The state of extreme deflection, when the velocities are zero, may be regarded as a result of the static action of the forces F_i. If these forces were known, it would be possible to find the exact mode of vibration and subsequently the natural frequency. However, these forces, as is seen from the formula, depend themselves on the frequency and are not known in advance; yet the problem can be solved by making successive approximations according to one of the following schemes.

First scheme of calculations. Taking the static-deflection curve as the mode of vibration, find a first approximation to the natural frequency p_{I} by formula (4.30) Next calculate the inertia forces for the first approximation

$$F_{i\mathrm{I}} = m_i a_{i\mathrm{I}} p_{\mathrm{I}}^2.$$

Taking them as the load, determine the corresponding displacements $a_{i\mathrm{II}}$ and then find a second approximation to the frequency by formula (4.30)

$$p_{\mathrm{II}}^2 = \frac{\Sigma F_{i\mathrm{I}} a_{i\mathrm{II}}}{\Sigma m_i a_{i\mathrm{II}}^2}.$$

Since all the forces $F_{i\mathrm{I}}$ can be changed simultaneously in the same ratio (this will not affect the value of p^2), they can be taken equal to $m_i a_{iI}$ leaving out the factor p_I^2 common to all the forces F_{iI}. Further the process is repeated to determine the inertia forces for the second approximation

$$F_{iII} = m_i a_{iII}$$

(the quantity p_{II}^2 can be left out, as before), the corresponding displacements a_{iIII} and a third approximation to the frequency

$$p_{III}^2 = \frac{\Sigma F_{iII} a_{iIII}}{\Sigma m_i a_{iIII}^2} .$$

The computations are continued until two successive values are found to be sufficiently close.

Second scheme of calculations. An alternate method suggested by Stodola consists in the following. Assign a first approximation to the deflection curve a_{iI} and write the corresponding inertia forces

$$F_{iI} = m_i a_{iI} p_I^2,$$

where p_I^2 is as yet unknown. Find the deflections a_{iII} due to the forces F_{iI} reduced by a factor of p_I^2 (i.e., due to the forces $m_i a_{iI}$); the actual deflections due to the forces F_{iI} will be p_I^2 times as large, i.e., equal to $p_I^2 a_{iII}$. Clearly, the curve defined by the deflections a_{iII} would be similar to the original curve a_{iI} if the exact shape of the latter were assigned; in this case the following equality should be fulfilled: $a_{iI} = p_I^2 a_{iII}$. Hence we find

$$p_I^2 = \frac{a_{iI}}{a_{iII}}$$

However, since the mode of vibration has been assigned approximately, the result will also be approximate, with the ratio a_{iI}/a_{iII} differing for different points i (because the a_{iI}'s are assigned approximately); therefore, the approximate value of the square of the frequency is generally taken to be the value corresponding to the section of maximum deflection

$$p_I^2 = \frac{a_{I\,\max}}{a_{II\,\max}} .$$

Thus, the approximate value of the square of the frequency is equal to the ratio of the characteristic ordinates of two curves: the deflection curve a_{iI} (to be assigned) and the calculated deflection curve a_{iII} due to the loads $m_i a_{iI}$. If the ratios a_{iI}/a_{iII} are practically the same for all numbers i, we may be content with the result obtained. When the ratios a_{iI}/a_{iII} depend markedly on i, the next approximation should be made: taking the mode a_{iII} as the starting one, determine the forces $m_i a_{iII}$ and the deflections a_{iIII}. A second approximation to the square of the frequency is then

$$p_{\mathrm{II}}^2 = \frac{a_{\mathrm{II\,max}}}{a_{\mathrm{III\,max}}}.$$

The process is repeated until the ratios $a_{k\,\max}/a_{k+1\,\max}$ are the same to a sufficient degree of accuracy.

Decomposition into Simpler Systems

In contrast to Rayleigh's method the methods to be developed below give too low values of the frequency. Therefore, they may be used to advantage in conjunction with the energy method; two-sided estimates are then obtained for the true value of the frequency.

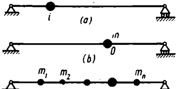


Fig. 26

Method of decomposing masses. Assume that there is a concentrated mass m_i at a certain point of an elastic system (Fig. 26*a*) and the natural frequency of the system is

$$p_i = \sqrt{\frac{k_i}{m_i}}.$$

At some different point O, chosen as the point of reduction (Fig. 26*b*), place a mass m so that the natural frequency of the new system $\sqrt{k_0/m}$ will be equal to the frequency p_i. Equating the frequencies, we find the reduced mass

$$m = \frac{k_0}{k} m_i. \tag{4.42}$$

Thus, the transfer of the mass m_i to point O will leave the frequency unchanged if the mass being transferred is multiplied by the reduction factor k_0/k_i.

We extend this procedure to the case when the system contains several masses, $m_1, m_2, \ldots, m_n$ (Fig. 26*c*). To each of these masses corresponds an appropriate reduction factor, as is seen from formula (4.42). The total reduced mass is

$$m_0 = \frac{k_0}{k_1} m_1 + \frac{k_0}{k_2} m_2 + \ldots + \frac{k_0}{k_n} m_n. \tag{4.43}$$

Dividing both sides of the above equality by k_0 and noting that $m_0/k_0 = 1/p^2$, where p^2 is the square of the frequency of the reduced system, we obtain *Dunkerley's approximate formula*

$$\frac{1}{p^2} = \frac{1}{p_1^2} + \frac{1}{p_2^2} + \ldots + \frac{1}{p_n^2}. \tag{4.44}$$

Though the transfer of a single mass according to formula (4.42) is undoubtedly legitimate, the extension of this procedure to several masses is not rigorous and is based on the assumption that these masses have no influence on each other. Therefore, the frequency p^2 as determined by formula (4.44) does not give the exact solution to the problem.

However, in most cases this formula, being very simple, provides acceptable accuracy. To determine the natural frequency it is necessary only to calculate the frequencies $p_1, p_2, \ldots, p_n$ each of which relates to a "particular" system of one degree of freedom; and the point of reduction need not be chosen and fixed, which makes the solution completely determinate.

When there is a distributed mass $m = m(x)$ in place of concentrated masses, formula (4.43) becomes

$$m_0 = k_0 \int_0^l \frac{m(x)\,dx}{k(x)}$$

and, consequently,

$$\frac{1}{p^2} = \int_0^l \frac{m(x)\,dx}{k(x)},$$

where $k(x)$ is the spring stiffness corresponding to the point with abscissa x.

Based on Rayleigh's theorem, it may be verified that Dunkerley's formula always gives too low values for the frequency.

Consider the first "particular" system containing a single mass m_1 and determine the natural frequency by formula (4.27). If the mode of vibration corresponding to the single-mass system is substituted into this formula, the exact value of the frequency p_1 will be obtained. Assume, however, that the mode of vibration of the given multimass system is substituted into this formula; the formula will then give the result exceeding the true value of the square of the frequency p_1

$$\frac{\int_0^l EI\,(f'')^2\,dx}{m_1 a_1^2} > p_1^2.$$

Further, analogous inequalities can be written for the other "particular" systems

$$\frac{\int_0^l EI\,(f'')^2\,dx}{m_2 a_2^2} > p_2^2, \quad \frac{\int_0^l EI\,(f'')^2\,dx}{m_3 a_3^2} > p_3^2, \quad \ldots .$$

Inverting the fractions, we find

$$\frac{m_1 a_1^2}{\int\limits_0^l EI\,(f'')^2\,dx} < \frac{1}{p_1^2},\ \frac{m_2 a_2^2}{\int\limits_0^l EI\,(f'')^2\,dx} < \frac{1}{p_2^2},\ \dots .$$

Add the left-hand and the right-hand sides of the resulting inequalities

$$\frac{\Sigma m_i a_i^2}{\int\limits_0^l EI\,(f'')^2\,dx} < \frac{1}{p_1^2} + \frac{1}{p_2^2} + \dots + \frac{1}{p_n^2}.$$

Since $f(x)$ is the true mode of vibration of the given multimass system, according to formula (4.27) the left-hand side is exactly equal to $\frac{1}{p_0^2}$, where p_0 is the true frequency; consequently,

$$\frac{1}{p_0^2} < \frac{1}{p_1^2} + \frac{1}{p_2^2} + \dots + \frac{1}{p_n^2}.$$

Comparing this inequality with formula (4.44), we see that $p < p_0$, as stated above.

Example 5. Using Dunkerley's formula find the frequency for the beam represented in Fig. 23.

Consider the given system three times taking into account one mass at a time in succession. For the first scheme (see Table 1) we find

$$p_1^2 = \frac{3 \times 1{,}296EI}{25ml^3} = 155.5\,\frac{EI}{ml^3}.$$

The same result is true for the third scheme

$$p_3^2 = 155.5\,\frac{EI}{ml^3}.$$

For the second scheme

$$p_2^2 = \frac{48EI}{ml^3}.$$

Using formula (4.44) we calculate the reciprocal of the square of the frequency

$$\frac{1}{p^2} = \frac{ml^3}{155.5EI}\,2 + \frac{ml^3}{48EI} = 0.03368\,\frac{ml^3}{EI}.$$

Consequently,

$$p = \sqrt{\frac{EI}{0.03368ml^3}} = \frac{5.449}{l}\sqrt{\frac{EI}{ml}};$$

this result differs from the exact value by 5.5 per cent and involves an error of opposite sign, as compared with all the results obtained by Rayleigh's formula. Based on the results of the two obviously approximate calculations by formulas (4.27) and (4.44), it may be stated that the true value of the frequency lies within relatively narrow limits

$$\frac{5.449}{l}\sqrt{\frac{EI}{ml}} < p < \frac{5.696}{l}\sqrt{\frac{EI}{ml}}.$$

Method of decomposing restoring forces. This method was proposed for the case of transverse vibrations of a bar subjected to tensile forces.

In some practical cases (turbine blades, propeller blades, etc.) a bar executing transverse vibrations undergoes a simultaneous action of a distributed longitudinal load. In these cases the potential energy is due not only to the bending deformation of the beam but also to the longitudinal forces (Fig. 27)

$$U_{\max}=\frac{1}{2}\int_0^l [EI\,(f'')^2+N\,(f')^2]\,dx,$$

where $N = N(x)$ is the longitudinal force at the section of the bar.

The approximate value of the natural frequency can be obtained by the energy method using the formula

$$p^2=\frac{\int_0^l EI\,(f'')^2\,dx}{\int_0^l mf^2\,dx}+\frac{\int_0^l N\,(f')^2\,dx}{\int_0^l mf^2\,dx}. \qquad (4.45)$$

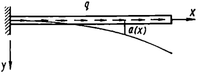


Fig. 27

Substituting a suitable expression for the mode of vibration, we obtain an approximate value of the frequency that is too high. However, formula (4.45) enables one to construct a different approximate solution giving a too low value of the frequency.

Consider the first term on the right-hand side of formula (4.45). It determines the square of the frequency of vibration of the bar with no longitudinal load, p_1^2. If the true mode of vibration of the beam subjected to the longitudinal forces is substituted into this expression, the result will be too high, as is always the case when using the energy method

$$\frac{\int_0^l EI\,(f'')^2\,dx}{\int_0^l mf^2\,dx}>p_1^2. \qquad (4.46)$$

Consider the second term on the right-hand side of formula (4.45), which corresponds to the case when the bar has no flexural rigidity; this term determines the square of the frequency of vibration of an inextensible string. Using the corresponding exact mode of vibration, we obtain the exact value of the square of the frequency of vibration of the string, p_2^2. With any other mode $f(x)$ the resulting frequency will be too high; if this mode is again the true mode of vibration of the given beam, the following inequality will be fulfilled

$$\frac{\int_0^l N(f')^2\,dx}{\int_0^l mf^2\,dx} > p_2^2. \tag{4.47}$$

It should be emphasized that $f(x)$ in inequalities (4.46) and (4.47) is one and the same function, the true mode of vibration of the given system. Adding together the left-hand and the right-hand sides of the two inequalities, we obtain

$$\frac{\int_0^l EI(f'')^2\,dx + \int_0^l N(f')^2\,dx}{\int_0^l mf^2\,dx} > p_1^2 + p_2^2.$$

According to formula (4.45) the left-hand side of this inequality is equal to the square of the true frequency p_0^2, since $f(x)$ is the true mode of vibration, and hence $p_0^2 > p_1^2 + p_2^2$. The sum $p_1^2 + p_2^2 = p^2$ can be taken as an approximate value of the square of the frequency; as is seen, it is less than the true value.

Thus, the determination of the natural frequency of a stretched bar calls first for the calculation of the frequencies for two "particular" systems: an inextensible beam and a flexible string. The first problem can be solved by any of the foregoing methods; the solution of the second problem is very simple if we take into account that the exact mode of vibration of a string is a straight line

$$f(x) = a\frac{x}{l},$$

where a is the displacement of the end of the string.

Then

$$f'(x) = \frac{a}{l}$$

and

$$p_2^2 = \frac{\int_0^l N\left(\frac{a}{l}\right)^2 dx}{\int_0^l ma^2\left(\frac{x}{l}\right)^2 dx} = \frac{\int_0^l N\,dx}{\int_0^l mx^2\,dx}.$$

If the longitudinal forces N result from the rotation of a bar of uniform mass distribution (Fig. 28), then

$$q = m\omega^2 x,$$

$$N = \int_x^l q\,dx = \frac{m\omega^2}{2}(l^2 - x^2),$$

$$\int_0^l N\,dx = \frac{m\omega^2 l^3}{3},$$

$$\int_0^l mx^2\,dx = \frac{ml^3}{3}.$$

Consequently, $p_2^2 = \omega^2$; this leads to the *Lamb-Southwell formula*

$$p^2 = p_1^2 + \omega^2, \tag{4.48}$$

i.e., the square of the natural frequency of a rotating bar is approximately equal to the sum of the square of the natural frequency calculated without regard to rotation and the square of the angular velocity (the resultant frequency is too low).

This formula is also valid in the case shown in Fig. 28 since

$$p_2^2 = \frac{\int_0^l N\left(\frac{a}{l}\right)^2 dx}{ma^2} = \frac{N}{ml},$$

but $N = m\omega^2 l$, which leads again to formula (4.48).

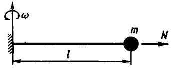


Fig. 28

Example 6. Compare the results obtained for the schematic diagram shown in Fig. 28 by the Lamb-Southwell formula, the energy method and the spring-constant formula (see Item 10, Table 1).

Using the Lamb-Southwell formula we find

$$p^2 = \frac{3EI}{ml^3} + \omega^2.$$

In the energy method, we assume for $f(x)$ the deflection curve due to a uniformly distributed load

$$f(x) = f_0\left[\frac{3}{2}\left(\frac{x}{l}\right)^2 - \frac{1}{2}\left(\frac{x}{l}\right)^3\right].$$

We then obtain

$$\int_0^l EI\,(f'')^2\,dx = \frac{3EIf_0^2}{l^3},$$

$$\int_0^l N\,(f')^2\,dx = m\omega^2 l \int_0^l (f')^2\,dx = \frac{6}{5}m\omega^2 f_0^2.$$

Using formula (4.45) we find (the result is too high)

$$p^2=\frac{\frac{3EIf_0^2}{l^3}+\frac{6}{5}m\omega^2 f_0^2}{mf_0^2}=\frac{3EI}{ml^3}+1.20\omega^2.$$

The true value of the natural frequency lies in a relatively narrow interval

$$\frac{3EI}{ml^3}+\omega^2<p_0^2<\frac{3EI}{ml^3}+1.2\omega^2.$$

The exact solution of the problem can be obtained by the use of the spring-constant formula

$$k=\frac{\alpha^3 EI\cosh\alpha l}{\alpha l\cosh\alpha l-\sinh\alpha l}$$

with

$$\alpha=\sqrt{\frac{N}{EI}}=\omega\sqrt{\frac{ml}{EI}}.$$

The natural frequency is then determined from the formula $p_0^2=k/m$.

5. Effect of Inelastic Resisting Forces on Free Vibrations of Linear Systems of One Degree of Freedom

General Considerations

So far it has been assumed that no energy is dissipated in vibration and the process of free vibration is of an undamped type. Experience shows, however, that the vibrations of an elastic system induced by a single disturbance are gradually damped out. The cause of damping is that *dissipative inelastic resisting forces* develop as well as elastic forces during vibration. Energy is being continuously and irreversibly expended in overcoming inelastic resistances, with the result that the total energy reserve gradually diminishes and the amplitudes of vibration decrease.

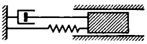


Fig. 29

To describe inelastic resisting forces it is customary to use the following simplified representations, according to the nature of these forces (see also Fig. 8).

Viscous resistance. It is a resistance proportional to the velocity. If, as usual, the velocity is denoted by v, the viscous-damping force is defined by the expression

$$R=cv, \tag{5.1}$$

where c is the proportionality factor.

An example is provided by a system with a hydraulic shock absorber (Fig. 29) which produces a resistance to the motion of a piston depending not on the displacement (as in the case of elastic constraints) but on the velocity and proportional to its first power. Such devices are employed, for instance, in automobile suspension designs. The hydraulic damper consists of one or several cylinders with pistons or of a chamber with a rotating impeller. The cylinders and chamber are filled with a damping liquid. As the pistons or impeller move, the liquid is forced through sized orifices; this produces a resistance of a nearly viscous type.

In formula (5.1) R implies the *force acting on the damper*; the viscous reaction of the dampers on the vibrating body has the opposite sense.

Resistance proportional to the nth power of the velocity. In some engineering devices, such as hydraulic dampers, the dependence of inelastic resisting forces on the velocity may be stronger. Such forces are often written as

$$R = cv\,|v|^{n-1}, \qquad (5.2)$$

where c is a constant.

Coulomb friction. In this case $n = 0$, i.e., the inelastic resisting force is assumed to be constant in magnitude but opposite in direction to the velocity

$$R = \pm c = cv\,|v|^{-1}.$$

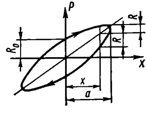


Fig. 30

Hysteresis. For mechanical systems subjected to cyclic loading, the force-displacement curve deviates from Hooke's law due to inelastic resistances and assumes the form of a hysteresis loop (Fig. 30). If to a displacement x corresponds an elastic resisting force P from Hooke's law, the total force will be greater or less than P, depending on the direction of motion (the sign of the velocity). During loading the force will be $P + R$ (the upper branch of the loop) and during unloading it will be $P - R$ (the lower branch of the loop); in Fig. 30 the arrowheads indicate loading and subsequent unloading.

In real mechanical systems the hysteresis phenomena are due to the internal friction in the material and, mainly, the friction in fixed connections (press fits, bolted joints, riveted joints, etc.) because of the slipping between the surfaces in contact. The intensity of this kind of inelastic resistance is indicated by the area inside the hysteresis loop which determines the energy dissipated in the structure during one loading-unloading cycle.

Based on an analysis of numerous experimental data, it has been found that for a number of structural materials the velocity has no effect on the area Ψ of the hysteresis loop and this area may be represented as

$$\Psi = ka^{n+1}, \tag{5.3}$$

where k and n = constants depending on the type of structure and material,

a = amplitude of vibration.

Consider successively the processes of damped free vibration for various types of inelastic resistance.

Free Vibrations with Viscous Resistance

Consider the diagram shown in Fig. 29. If the viscous-damping forces are taken into account, the differential equation of motion of the mass (Fig. 31) assumes the form

$$-kx - c\dot{x} = m\ddot{x}.$$

Denote $c/2m = n$, $k/m = p^2$; here the coefficient n characterizes the viscosity of the system (not to be confused with the coefficient n on p. 57). Now write the differential equation in standard form

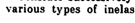


Fig. 31

$$\ddot{x} + 2n\dot{x} + p^2x = 0. \tag{5.4}$$

When $p^2 > n^2$, as is usually the case, the general solution of Eq. (5.4) can be represented as

$$x = ae^{-nt}\sin(\sqrt{p^2 - n^2}\,t + \alpha). \tag{5.5}$$

The constants a and α are determined by the initial conditions. In place of expressions (4.4a) we obtain

$$a = \sqrt{x_0^2 + \frac{(v_0 + nx_0)^2}{p^2 - n^2}},$$

$$\alpha = \arctan\frac{x_0\sqrt{p^2 - n^2}}{v_0 + nx_0}.$$

Solution (5.5) can also be rewritten in a form similar to expression (4.4)

$$x = e^{-nt}\left(x_0\cos\sqrt{p^2 - n^2}\,t + \frac{v_0 + nx_0}{\sqrt{p^2 - n^2}}\sin\sqrt{p^2 - n^2}\,t\right). \tag{5.5a}$$

The vibration curve is represented in Fig. 32, where the damped nature of the process is clearly seen.

As is apparent from expression (5.5), the frequency of the vibration is given by the formula

$$p_* = \sqrt{p^2 - n^2} \tag{5.5b}$$

and generally differs only slightly from the frequency of undamped vibration of the same system with no damping.

Consider the successive amplitudes corresponding to the instants when $\sin(\sqrt{p^2-n^2}\,t+\alpha)$ becomes equal to unity

$$a_1 = ae^{-nt_1},\ a_2 = ae^{-n(t_1+T)},\ a_3 = ae^{-n(t_1+2T)},\ \ldots,$$

where t_1 is the time corresponding to the first maximum deflection; $T = 2\pi/\sqrt{p^2-n^2}$ is the period of vibration (it should be borne in mind that this term is conventional as, strictly speaking, the motion is not periodic).

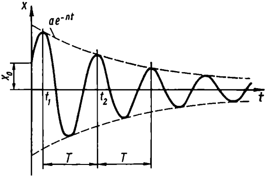


Fig. 32

The ratio between any two consecutive amplitudes (*decrement*) remains constant at all times

$$\frac{a_1}{a_2} = \frac{a_2}{a_3} = \ldots = e^{nT},$$

i.e., the sequence of amplitudes forms a geometric progression. Consequently, for any value of i the following equality holds

$$\delta = nT = \ln\frac{a_i}{a_{i+1}}. \qquad (5.6)$$

Expression (5.6) determines the rate of damping and is called the *logarithmic decrement.*

Example 7. For a vibrating elastic system, it is found that the amplitude is reduced one half for each cycle. Determine the logarithmic decrement and the change in natural frequency due to damping.

Using formula (5.6) we find

$$\delta = nT = \ln 2 = 0.693,$$

whence

$$n=\frac{0.693}{T}=\frac{0.693}{2\pi}\sqrt{p^2-n^2}.$$

Solving this equation we find that n^2 is very small in comparison with p^2

$$n^2=0.012p^2.$$

The natural frequency of vibration

$$p_*=\sqrt{p^2-n^2}=\sqrt{p^2-0.012p^2}=0.994p$$

differs from the frequency of the corresponding undamped vibration by only 0.6 per cent.

The effect of a small amount of viscous damping on the frequency is negligible; at the same time even a slight resistance damps free vibrations intensively. This makes it possible, on the one hand, to ignore viscous damping in the calculation of the frequency and, on the other hand, to consider the free vibrations practically vanished after a certain length of time. In the above example the amplitude is only $(0.5)^{10}\cong 0.001$ of its initial value after 10 cycles.

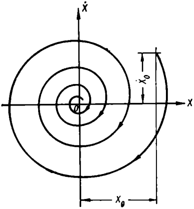


Fig. 33

In order to construct the *phase diagram* of a damped system it is necessary to consider the set of equations

$$x=ae^{-nt}\sin(p_*t+\alpha),$$
$$\dot{x}=ae^{-nt}[p_*\cos(p_*t+\alpha)-n\sin(p_*t+\alpha)]$$

as the equation of the phase trajectory in parametric form.

A typical phase trajectory is shown in Fig. 33. It represents a spiral wound around the origin. The phase diagram is formed by a system of such spirals surrounding the singular point O; the latter is called the *stable focus* in this case.

Free Vibrations with a Resistance Proportional to the n*th Power of the Velocity*

In this case, in place of Eq. (5.4) we obtain a non-linear differential equation of motion in the form

$$m\ddot{x}+c\dot{x}\,|\dot{x}|^{n-1}+kx=0;$$

this equation cannot be solved in closed form. For practical purposes use may be made of the approximate *method of energy balance*.

The idea of this method is to equate the energy lost by the system to the work done by the inelastic resisting force during one cycle

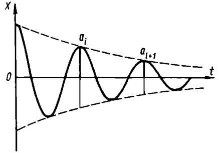


Fig. 34

of vibration; the two quantities are expressed approximately (accurate determination would require knowledge of the law of motion).

Let us first estimate the change in energy for one cycle (Fig. 34). The potential energy at two consecutive positions of maximum deflection (points i and $i+1$) is

$$U_i=\frac{1}{2}ka_i^2,\ U_{i+1}=\frac{1}{2}ka_{i+1}^2,$$

where k is the spring constant of the system. The increment (negative) of energy for the cycle under consideration is

$$\Delta U_i=\frac{k}{2}(a_{i+1}^2-a_i^2).$$

For moderate damping the latter expression can be written in a simpler form

$$\Delta U_i = \frac{k}{2}(a_{i+1}+a_i)(a_{i+1}-a_i) \cong k a_i \Delta a_i, \tag{5.7}$$

where $\Delta a_i = a_{i+1} - a_i$ represents the increment (negative) of the amplitude for one cycle. Referring to Fig. 34 it may be taken

$$\Delta a = T\frac{da}{dt}, \tag{5.8}$$

where T is the period of the vibration which may be considered coinciding with the period of free vibration of the undamped system. Leaving out the subscripts, we finally have

$$\Delta U = kaT\frac{da}{dt}. \tag{5.9}$$

Determine now the work done per cycle by the inelastic resisting force $-R = -c\dot{x}|\dot{x}|^{n-1}$; assume that during this cycle the vibrations are defined approximately by a sinusoidal relation

$$x = a\cos pt \tag{5.10}$$

(it is assumed here that the frequency of damped vibration is no different from the natural frequency of vibration of the undamped system).

The expression for the work is

$$\Psi = -\int_0^T R\dot{x}\,dt = -\int_0^T c\dot{x}^2|\dot{x}|^{n-1}\,dt = -c\int_0^T |\dot{x}|^{n+1}dt.$$

Substituting according to relation (5.10)

$$\dot{x} = -ap\sin pt,$$

we obtain

$$\Psi = -ca^{n+1}p^{n+1}\int_0^T |\sin pt|^{n+1}\,dt$$

or

$$\Psi = -ca^{n+1}p^{n}S, \tag{5.11}$$

where the integral

$$S = \int_0^{2\pi} |\sin\gamma|^{n+1}\,d\gamma \tag{5.12}$$

depends only on the power n

n.....	0	0.5	1.0	1.5	2	2.5	3
S.....	4.000	3.500	3.142	2.874	2.666	2.498	2.356

Equating the expressions for ΔU and Ψ [formulas (5.9) and (5.11)], we obtain the differential equation for the upper envelope

$$-ba^n = \frac{da}{dt}, \tag{5.13}$$

where the coefficient

$$b = \frac{cp^{n+1}S}{2\pi k} \tag{5.14}$$

depends only on the parameters of the system.

Consider the case when $n = 1$, i.e., the case of *viscous resistance*. In place of Eq. (5.13) we have

$$-ba = \frac{da}{dt}; \tag{5.15}$$

expression (5.14) becomes in this case

$$b = \frac{c}{2m}.$$

Separating the variables in formula (5.15)

$$-b\,dt = \frac{da}{a}$$

and integrating from $t = 0$ to t, we find

$$\ln\frac{a}{a_0} = -bt,$$

i. e.,

$$a = a_0e^{-bt} = a_0e^{-\frac{ct}{2m}}.$$

We have thus obtained the earlier expression for the damped-vibration envelope.

Let us turn to the case when $n \neq 1$. Separating the variables in Eq. (5.13), we have

$$-b\,dt = \frac{da}{a^n}.$$

Integrating between the limits $(0, t)$ we obtain

$$\frac{a^{1-n}}{1-n} - \frac{a_0^{1-n}}{1-n} = -bt,$$

whence

$$a = \frac{a_0}{\sqrt[n-1]{1+(n-1)\,bta_0^{n-1}}} \tag{5.16}$$

This expression determines the law of damping of free vibration for any value of n different from unity. Of course, this solution cannot provide any details of the process of damped vibration $x(t)$ but such details are unimportant.

Consider two special cases.

Case $n = 0$ (*dry friction*). In this case we obtain from expression (5.16)

$$a = a_0 - bt, \qquad (5.17)$$

i.e., the envelope is a straight line. According to formula (5.14), for $n = 0$

$$b = \frac{4c}{kT}; \qquad (5.18)$$

this expression determines the rate of damping.

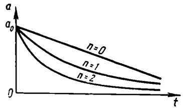


Fig. 35

Case $n = 2$ (*hydrodynamic or turbulent damping*). From the fundamental expression (5.16) we have the equation of the envelope

$$a = \frac{a_0}{1 + ba_0t}.$$

i.e., the envelope is a hyperbola

Figure 35 shows schematically the envelopes for the cases $n = 0, 1, 2$.

Free Vibrations with Coulomb Friction

In the case of dry friction the equation of motion is written as

$$m\ddot{x} + kx \pm c = 0;$$

the sign of the last term is to be chosen according to the sign of the velocity (the direction of motion). Let, for example, the motion be started at a time $t = 0$ when $x = a_0$, $x = 0$. Then the velocity is negative in the first interval of motion and the minus sign should be adopted in the equation.

Denoting $\alpha = c/k$ and $p^2 = k/m$, we obtain the equation

$$\ddot{x} + p^2x - p^2\alpha = 0.$$

The solution of this equation with the above initial conditions is

$$x = (a_0 - \alpha)\cos pt + \alpha. \qquad (5.19)$$

For the velocity we obtain the expression

$$\dot{x} = -(a_0 - \alpha)\, p \sin pt.$$

When the argument pt becomes equal to π, the velocity vanishes again, i.e., the system reaches its extreme deflection on the other side of the origin; this deflection, according to Eq. (5.19), is

$$a_1 = (a_0 - \alpha)\cos\pi + \alpha = -a_0 + 2\alpha,$$

i.e., it is 2α less, numerically, than the initial deflection.

If the absolute value of a_1 satisfies the inequality $k\,|a_1| > c$ (or $|a_1| > \alpha$), the elastic force is larger than the friction force

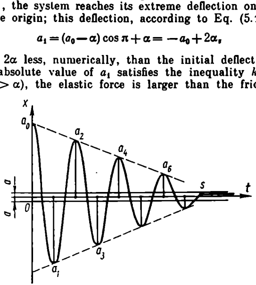


Fig. 36

and the system starts moving in the direction of positive values of x. The equation of motion is now written as

$$\ddot{x} + p^2 x + p^2\alpha = 0.$$

Shifting the origin of time, we assume the initial conditions: $x = a_1$ and $v = 0$ at $t = 0$. The solution is then written as

$$x = (a_1 + \alpha)\cos pt - \alpha.$$

Reasoning as above, we obtain for the next deflection

$$a_2 = -a_1 - 2\alpha$$

or, expressing a_1 in terms of a_0,

$$a_2 = a_0 - 4\alpha.$$

Thus, the amplitude diminishes the same amount $4\alpha = 4c/k$ for each cycle, i.e., the sequence of amplitudes forms an arithmetic progression and the envelope of the curve $x(t)$ is a straight line. The tangent of the angle that the line makes with the t axis is $4c/kT$.

Calculations can be continued only until the inequality $|a_i| > \alpha$ is fulfilled. As soon as the deflection a_i becomes less than α, the motion ceases completely since the elastic force ka_i will not be large enough to resist the friction force.

The graph of the vibration is given in Fig. 36. It consists of segments of sine curves with the same period but with different amplitudes. Two horizontal lines $x = \pm\alpha$ determine the "dead" zone: if the velocity becomes zero within this zone, the motion stops (point s).

The same results could have been obtained immediately by treating the dry-friction force as a special case of the force R in expression (5.2) if $n = 0$

$$R = cv\,|v|^{-1}. \tag{5.20}$$

Indeed, the product $v\,|v|^{-1}$ is equal to $+1$ or -1 depending on the sign of the velocity v. Therefore the result (5.17) derived for $n = 0$ applies directly to the case of dry friction. That is why the slope of the envelope $4c/kT$ found here is identical with expression (5.18).

Free Vibrations with Internal Friction

As stated above, the energy dissipated per cycle is independent of the rate of cyclic straining and its absolute value is given by formula (5.3). On the other hand, this energy is expressed by formula (5.9). Equating $\Delta U = -\Psi$, we obtain the differential equation for the upper envelope of the damped-vibration curve

$$-\frac{c}{kT}a^n = \frac{da}{dt}.$$

As in the case of an inelastic resistance proportional to the nth power of the velocity, the solution of this differential equation provides no detailed information on the process of damped vibration but it enables the envelope to be found easily.

Putting $b = c/kT$, we arrive at the previously solved differential equation (5.13) but with a different value of the parameter b. The final result in this case is of the same form as (5.16); thus,
for $n = 0$

$$a = a_0 - bt = a_0 - \frac{ct}{kT};$$

for $n = 1$

$$a = a_0 e^{-bt} = a_0 e^{-\frac{ct}{kT}};$$

for $n=2$

$$a=\frac{a_0}{1+ba_0t}=\frac{a_0}{1+\frac{ca_0t}{kT}}.$$

The envelopes corresponding to these cases are shown schematically in Fig. 35.

It is important to note that identical records of damped vibration may be obtained for different inelastic resisting forces.

Suppose that the amplitudes diminish according to a law of geometric progression, i.e., the envelope is an exponential curve, as revealed from the test record of vibration. It does not follow, however, that the damping of vibration is due to viscous friction. The same type of damping may be caused by hysteresis when energy losses due to internal friction are totally independent of the velocity.

In order to ascertain the true nature of friction forces, the record of vibration alone is insufficient. Some additional experiments should be made.

Generalization of the Concept of Logarithmic Decrement

The concept of the logarithmic decrement given above can be extended to any processes of damped vibration assuming that in all cases the logarithmic decrement δ is the natural logarithm of the ratio between two consecutive amplitudes. This ratio is generally variable so that the logarithmic decrement is not a constant number for the whole process of vibration but gradually varies; exceptions are the cases of viscous resistance and hysteresis (for $n = 1$) when δ = constant for the whole process of vibration.

Thus, we assume

$$\delta=\ln\frac{a_i}{a_{i+1}}.$$

If the damping is moderate, then a_i/a_{i+1} differs only slightly from unity and we may assume

$$\ln\frac{a_i}{a_{i+1}}\cong\frac{a_i}{a_{i+1}}-1=\frac{a_i-a_{i+1}}{a_{i+1}};$$

consequently.

$$\delta=\frac{\Delta a}{a}. \tag{5.21}$$

This expression can be given another interpretation. The energy dissipated per cycle is defined by formula (5.7); the maximum potential energy is

$$U=\frac{ka^2}{2}.$$

Setting up the ratio between the two energies, we obtain a quantity twice as large as (5.21)

$$\frac{\Delta U}{U}=2\,\frac{\Delta a}{a}.$$

Hence the following definition: for moderate amounts of damping the logarithmic decrement is the ratio of the energy dissipated per cycle to twice the maximum potential energy for the cycle.

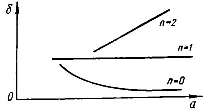


Fig. 37

Taking into account expression (5.8), formula (5.21) may also be written as

$$\delta=-\frac{da}{dt}\,\frac{T}{a}. \qquad (5.22)$$

The differential equation (5.13) holds for all the foregoing cases; using it, we obtain the following relationship between the logarithmic decrement and the amplitude of vibration (Fig. 37)

$$\delta=bTa^{n-1}. \qquad (5.23)$$

Thus, in the case of $n=1$

$$\delta=bT,$$

i.e., the logarithmic decrement is constant.

For dry friction ($n=0$) the logarithmic decrement increases with decreasing amplitude

$$\delta=\frac{bT}{a}.$$

In the case of $n=2$

$$\delta=bTa.$$

i.e., the logarithmic decrement diminishes with decreasing amplitude.

Relation (5.23) can be used as the basis for the experimental determination of the parameter n.

6. Undamped Systems of One Degree of Freedom with Non-Linear Restoring Forces

Some Types of Non-Linear Characteristics

An example of a non-linear elastic constraint is provided by the rear-axle suspension of a car (Fig. 38a) where, besides the main spring, there is an additional spring (dual-rate spring). For small displace-

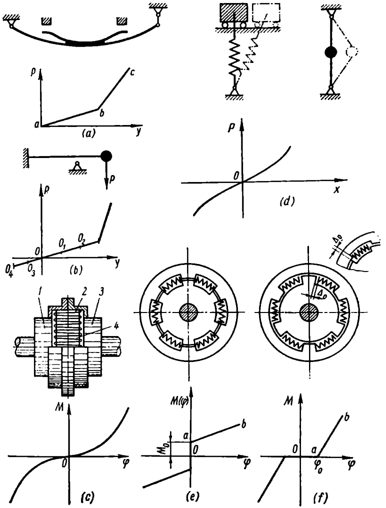

p
c
b
a
(a)
y
p
0
x
(d)
P
p
0
0_1
0_2
0_4
0_3
(b)
y
1
2
3
4
Δ_0
Δ_0
M
0
φ
(c)
M(φ)
a
b
M_0
0
φ
(e)
M
b
0
a
φ_0
φ
(f)

Fig. 38

ments of the body, the ends of the second spring do not touch the stops and only the main spring is working; the relation between the pressure P on the spring and its deflection y can be considered to be linear (portion ab). For large displacements of the body, the ends of the second spring press against the frame brackets and the combined spring becomes stiffer (portion bc). Thus, the overall spring characteristic $P(y)$ is non-linear.

Figure 38*b* shows a schematic diagram, which is basically equivalent to the automobile suspension, and the corresponding characteristic. The position of the zero point, about which the system is vibrating, depends on the static load; as the static load increases, the zero point moves successively to positions O_1, O_2, etc., and as the load decreases it moves to positions O_3, O_4, etc.

Figure 38*c* shows a non-linear coupling which serves to connect the two parts of a shaft. Threads *4* on half-couplings *1* and *3* are wrapped with a zigzag steel strip *2*. As the transmitted torque M increases, the strip tightens and the coupling becomes stiffer. The relation between the angle φ of relative rotation of the half-couplings and the torque M is shown in the same figure.

Similar characteristics are found in the systems shown in Fig. 38*d*. A gradual increase in stiffness is indicated by increasing steepness of the characteristic.

A schematic diagram of a non-linear spring coupling with an initial *compression* is shown in Fig. 38*e*. When the torque is small, the relative rotation of the half-couplings is zero. As the torque reaches a value M_0 (determined by the amount of the initial compression of the springs), the springs begin to deform, making it possible for the half-couplings to rotate with respect to each other. The characteristic of this coupling contains a vertical section coinciding with part of the axis of ordinates.

Figure 38*f* shows a coupling with *clearances* Δ_0 and its characteristic. Up to an angle $\varphi_0 = \Delta_0/2r$ the half-couplings can rotate freely with respect to each other and only at larger angles can the torque be transmitted.

Reliable spring characteristics can usually be obtained only experimentally from static tests, but for simple diagrams they can be constructed analytically by calculation.

Consider the construction of the characteristic for the system shown in Fig. 38*d*. We shall first assume that, when the mass is at mid-position, the spring is unstretched.

When the mass is displaced a distance x, the spring elongates an amount $\sqrt{x^2 + l^2} - l$ and the corresponding spring force is

$$N = k(\sqrt{x^2 + l^2} - l).$$

The horizontal component of this force, which determines the spring characteristic of the system, is

$$P = N\frac{x}{\sqrt{l^2+x^2}} = kx\left[1-\frac{1}{\sqrt{1+\frac{x^2}{l^2}}}\right].$$

Assuming that the displacement x is small in comparison with the length l, we can write

$$\frac{1}{\sqrt{1+\frac{x^2}{l^2}}} \cong 1-\frac{1}{2}\left(\frac{x}{l}\right)^2$$

and, consequently the restoring force is purely non-linear

$$P \cong \frac{kx^3}{2l^2}. \tag{6.1}$$

If the initial tension in the spring is N_0, the total spring force when the mass is deflected an amount x is

$$N = N_0 + k\left(\sqrt{l^2+x^2}-l\right)$$

and its horizontal component is

$$P = N\frac{x}{\sqrt{l^2+x^2}} \cong N_0\frac{x}{l}+\frac{kx^3}{2l^2}.$$

This expression differs from expression (6.1) by the presence of the linear term. From this it follows that the initial tension may have an appreciable effect on the spring characteristic of the system.

Let us construct the characteristic for the coupling with an initial compression (see Fig. 38*e*). If, when the coupling is subjected to a torque, each spring is acted on by a force smaller than the initial compressive force P_0, the relative rotation of the half-couplings is impossible. Denoting the number of springs by n and the radius of the boundary circle by r, we can write that as long as the torque is less than $M_0 = nP_0r$, the angle of relative rotation is zero (portion Oa in Fig. 38*e*). If the torque M becomes larger than M_0, the springs begin to deform; each spring will carry an additional force (above the initial force P_0) $M/nr - P_0$ and therefore will deflect an amount $(M/nr - P_0)/k$ (k is the stiffness of one spring in compression). The corresponding angle of relative rotation of the half-couplings is

$$\varphi = \frac{M/nr - P_0}{kr}.$$

Hence we find that for $M \geqslant M_0$

$$M = knr^2\varphi + nP_0r;$$

this relationship is shown in Fig. 38e (portion *ab*); in the portion under consideration the stiffness of the coupling is knr^2.

In a similar way we can construct the characteristic for the coupling with clearances (Fig. 38f). There is no torque if the angle of rotation is less than $\varphi_0 = \Delta_0/2r$ (portion *Oa* in Fig. 38f). Only for $\varphi > \varphi_0$ is the torque transmitted. To a torque M corresponds a compressive force of one spring M/nr and the compression of the spring by an amount $\Delta = M/knr$. Consequently, the additional angle of rotation (above the angle φ_0) is

$$\frac{\Delta}{r} = \frac{M}{knr^2}.$$

The total angle of rotation, measuring from the mid-position, is

$$\varphi = \varphi_0 + \frac{M}{knr^2}$$

and from this we find the equation of the spring characteristic for $\varphi > \varphi_0$

$$M = knr^2(\varphi - \varphi_0)$$

(portion *ab* in Fig. 38f).

Exact Solution

The differential equation of free vibration of a non-linear elastic system of one degree of freedom should contain a non-linear restoring force $P(x)$ instead of a linear restoring force $-kx$; the specific expression for $P(x)$ is determined by the spring characteristic. The fundamental differential equation of the problem becomes

$$m\ddot{x} + P(x) = 0.$$

For torsional systems, the mass m should be replaced by the moment of inertia J, the displacement x by the angle of rotation φ and the restoring force $P(x)$ by the restoring moment $M(\varphi)$.

Dividing the equation by the mass m, we obtain

$$\ddot{x} + f(x) = 0, \tag{6.2}$$

where

$$f(x) = \frac{P(x)}{m}. \tag{6.3}$$

Since the solution of Eq. (6.2) presents difficulties, the problem is generally restricted to the determination of the *frequency of free vibration* without going into the details of the process of vibration; this will suffice for many practical applications.

Expressing the acceleration $\ddot{x}$ as

$$\ddot{x}=\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=\frac{dv}{dx}v$$

(v is the velocity), we obtain in place of Eq. (6.2)

$$v\frac{dv}{dx}+f(x)=0.$$

Separating the variables, we find

$$v\,dv=-f(x)\,dx.$$

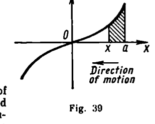


Fig. 39

To integrate we choose as the origin the time of maximum deflection when the displacement x is a maximum ($x_{\max}=a$) and the velocity is zero ($v=0$). Then

$$\int\limits_0^v v\,dv=-\int\limits_a^x f(x)\,dx$$

or

$$\frac{v^2}{2}=-\int\limits_a^x f(x)\,dx=\int\limits_x^a f(x)\,dx;$$

this relationship expresses the law of conservation of energy: the left-hand side represents the kinetic energy accumulated during the motion from the extreme position ($x=a$, $v=0$) to the current position (x, v) while the right-hand side represents the potential energy lost during the same motion [this energy is represented by the cross-hatched area in the graph of the non-linear restoring force characteristic (Fig. 39)]*. By taking the root, we find

$$v=\frac{dx}{dt}=-\sqrt{2\int\limits_x^a f(x)\,dx}. \qquad (6.4)$$

Of the two signs before the root the minus sign is taken because the velocity is negative in the interval of motion under consideration.

The integration of Eq. (6.4) gives the time t as a function of the displacement x

$$t=-\int\limits_a^x\frac{dx}{\sqrt{2\int\limits_x^a f(x)\,dx}}=\int\limits_x^a\frac{dx}{\sqrt{2\int\limits_x^a f(x)\,dx}}.$$

---

* Both energies are estimated per unit of mass.

If the integration is carried out from $x = 0$ to $x = a$, the time interval for a quarter cycle (quarter period) will be found for a system with a symmetric characteristic. Accordingly the period of the vibration is

$$T = 2\sqrt{2}\int_0^a \frac{dx}{\sqrt{\int_x^a f(x)\,dx}}; \tag{6.5}$$

this formula makes it possible to find the exact relation between the period of free vibration and the amplitude.

Consider the symmetric characteristic defined by the law

$$f(x) = \alpha x^{2n-1} \quad (n = 1,\ 2,\ \ldots).$$

We find successively

$$\int_x^a f(x)\,dx = \frac{\alpha}{2n}(a^{2n} - x^{2n}),$$

$$\int_0^a \frac{dx}{\sqrt{\frac{\alpha}{2n}(a^{2n} - x^{2n})}} = \sqrt{\frac{2n}{\alpha}}\,\frac{1}{a^{n-1}}\int_0^1 \frac{d\xi}{\sqrt{1-\xi^{2n}}}$$

(here $\xi = x/a$); and finally we obtain from formula (6.5)

$$T = 4\sqrt{\frac{n}{\alpha}}\,\frac{1}{a^{n-1}}\int_0^1 \frac{d\xi}{\sqrt{1-\xi^{2n}}}. \tag{6.6}$$

From this it is apparent that the period T is independent of the amplitude of vibration only for $n = 1$ (linear characteristic); in the other cases there exists a relationship between the period and amplitude.

Consider vibrations of a system with a cubic characteristic $f(x) = \alpha x^3$; then $n = 2$ and we find from expression (6.6)

$$T = \frac{4}{a}\sqrt{\frac{2}{\alpha}}\int_0^1 \frac{d\xi}{\sqrt{1-\xi^4}}.$$

The elliptic integral above can be evaluated using tables of special functions; its value is $1.8541/\sqrt{2}$. Consequently,

$$T = \frac{4}{a\sqrt{\alpha}}\,1.8541.$$

The frequency of free vibration is

$$p=\frac{2\pi}{T}=0.8472a\sqrt{\alpha}, \tag{6.7}$$

i.e., the frequency increases linearly with increasing amplitude.

It should be emphasized that the dependence (of one sort or another) of the frequency of free vibration on the initial conditions is typical of non-linear systems. For this reason the term "natural frequency" should be *avoided* when referring to non-linear systems.

In conclusion we note that the exact solution of the problem is easy to obtain in cases where the characteristic of the system is *piecewise linear*, i.e., consists of line segments. In such cases a series of separate stages of motion are considered for each of which the differential equation is linear; the constants of integration are found from the initial and matching conditions (equality of displacements and velocities in the transition from one state to another).

Let it be required, for instance, to analyse free vibrations of a system *with a clearance* when the characteristic is of the form

$$f(x)=\begin{cases} p_0^2(x-a_0) & \text{for } x>a_0,\\ 0 & \text{for } -a_0\leqslant x\leqslant a_0,\\ p_0^2(x+a_0) & \text{for } x<-a_0 \end{cases}$$

(see, for example, Fig. 38*f*). If the following conditions are specified at the initial instant

$$x(0)=a, \quad \dot{x}(0)=0,$$

then the differential equation (6.2) in the first stage of motion (for $x>a_0$) becomes

$$\ddot{x}+p_0^2(x-a_0)=0$$

having the solution

$$x=(a-a_0)\cos p_0t+a_0$$

and the velocity varies according to the law

$$\dot{x}=-(a-a_0)p_0\sin p_0t.$$

It follows that the co-ordinate x reaches the value a_0 at the time t_1 defined by the formula

$$t_1=\frac{\pi}{2p_0}.$$

At this instant the velocity is

$$\dot{x}(t_1)=-(a-a_0)p_0.$$

Referring now to the second stage, we write the differential equation as

$$\ddot{x}=0.$$

The solution of this equation must satisfy the matching conditions

$$x(t_1)=a_0, \quad \dot{x}(t_1)=-(a-a_0)\,p_0$$

giving

$$x=a_0-(a-a_0)\,p_0\,(t-t_1).$$

Hence the length of time t_2 in which the origin $x=0$ is reached is equal to

$$t_2=t_1+\frac{a_0}{(a-a_0)\,p_0}.$$

Clearly, the length of time t_2 determines the quarter period of the vibration so that the frequency of the vibration is given by the expression

$$p=\frac{2\pi}{T}=\frac{2\pi}{4t_2}=\frac{p_0}{1+\cfrac{2}{\pi\left(\cfrac{a}{a_0}-1\right)}}.$$

Representation of Motion in Phase Plane

To construct phase trajectories, we represent the fundamental differential equation of the problem (6.2) as two equations of the first order

$$\frac{dx}{dt}=v, \quad \frac{dv}{dt}=-f(x)$$

and divide the second equation by the first one. We then obtain the differential equation of phase trajectories

$$\frac{dv}{dx}=-\frac{f(x)}{v};$$

the time t is not involved in this equation. The set of integral curves of the equation forms the phase diagram of the system. Since the variables are separable, we obtain

$$v^2=-2\int f(x)\,dx+C.$$

The constant C is determined by one initial condition: $v=v_0$ at $x=x_0$, so that to each particular combination x_0, v_0 corresponds a distinct integral curve.

Let, for instance, $f(x) = \alpha x^3$; then

$$v^2 = -\frac{\alpha x^4}{4} + C. \tag{6.8}$$

To determine the constant C we substitute $x = x_0$ and $v = v_0$

$$C = \frac{\alpha x_0^4}{4} + v_0^2.$$

We now write Eq. (6.8) as

$$v^2 = \frac{\alpha}{4}(x_0^4 - x^4) + v_0^2,$$

i.e.,

$$v = \pm\sqrt{\frac{\alpha}{4}(x_0^4 - x^4) + v_0^2}.$$

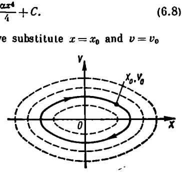


Fig. 40

A typical curve $v(x)$ which represents a phase trajectory for particular initial conditions is shown by a solid line in Fig. 40. When these conditions are varied, the phase trajectories assume forms such as are shown by dashed lines.

The general character of the phase diagram is no different from the one shown in Fig. 14 for a linear system; this similarity is due to the fact that both systems are conservative (linear elastic and non-linear elastic systems).

Approximate Solutions

Although formula (6.5) for the quarter period of free vibration is perfectly accurate, in practical problems it leads to cumbersome calculations which cannot be carried out in closed form. These difficulties can be avoided by using the following approximate methods for determining the frequency of free vibration.

The simplest method. The simplest (though very inaccurate) procedure is the following. Assume that the vibrations of a system are defined by the law

$$x = a\sin(pt + \alpha) \tag{6.9}$$

as is the case in linear systems. Expression (6.9) is the exact solution only if the characteristic $f(x)$ is linear, but in the general case the substitution of expression (6.9) in Eq. (6.2) does not reduce it to an identity. Let us ease the accuracy limitations and require that Eq. (6.2) be fulfilled at least at the instants when the deflection x reaches a maximum, i.e., equals a. The acceleration $\ddot{x}$ is then also a maximum

$$\ddot{x}_{\max} = -ap^2. \tag{6.10}$$

Consequently, at these instants the following equality must be fulfilled

$$-ap^2 + f(a) = 0,$$

i.e.,

$$p^2 = \frac{f(a)}{a}.$$

The last formula determines the frequency of free vibration p as a function of the amplitude a. Even though this formula is very inaccurate, it gives a correct general idea of the relation $a(p^2)$. Thus, if the non-linear characteristic is of the form

$$f(x) = p_0^2 x + \alpha x^3$$

(p_0, α are given numbers), then

$$p^2 = \frac{p_0^2 a + \alpha a^3}{a} = p_0^2 + \alpha a^2.$$

The graph of this relation is shown in Fig. 41 from which it is seen that the frequency increases for $\alpha > 0$ (in the case of a stiff characteristic) and decreases for $\alpha < 0$ (in the case of a soft characteristic) with increasing amplitude of free vibration.

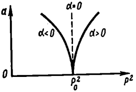


Fig. 41

In a similar manner, approximate relations $a(p^2)$ can be constructed for other types of characteristic.

Perturbation method. The basis of the method is the assumption that a system possesses "weak" non-linearity; such is, for example, the system whose vibrations are defined by the differential equation

$$\ddot{x} + p_0^2 x + \alpha x^3 = 0 \tag{6.11}$$

if the parameter α is small enough. In such cases the solution should be sought in the form of a power series in the small parameter

$$x = x_0 + \alpha x_1 + \alpha^2 x_2 + \ldots, \tag{6.12}$$

where x_0, x_1, x_2, ... are unknown functions of time which are to be determined.

In addition to expansion (6.12) it is also necessary to expand the coefficient p_0^2

$$p_0^2 = p^2 + \alpha c_1 + \alpha^2 c_2 + \ldots, \tag{6.13}$$

where p^2 is a new constant not yet known, c_1, c_2, ... are undetermined coefficients (suitable values are indicated below).

Substituting expressions (6.12) and (6.13) in Eq. (6.11) and limiting ourselves to those terms of the expansions which are written out, we obtain the equation

$$\ddot{x}_0+\alpha\ddot{x}_1+\alpha^2\ddot{x}_2+(p^2+\alpha c_1+\alpha^2 c_2)(x_0+\alpha x_1+\alpha^2 x_2)+ \\ +\alpha(x_0+\alpha x_1+\alpha^2 x_2)^3=0.$$

Opening the parentheses and retaining the terms containing α to a power not higher than the second, we have

$$\ddot{x}_0+p^2x_0+\alpha(\ddot{x}_1+p^2x_1+c_1x_0+x_0^3)+\alpha^2(\ddot{x}_2+p^2x_2+c_2x_0+ \\ +c_1x_1+3x_0^2x_1)=0.$$

This equation must hold for any small value of α. Therefore the coefficients of $\alpha^0, \ldots \alpha^1, \alpha^2, \ldots$ must vanish separately; this leads to the system

$$\left.\begin{aligned} &\ddot{x}_0+p^2x_0=0, \\ &\ddot{x}_1+p^2x_1=-c_1x_0-x_0^3, \\ &\ddot{x}_2+p^2x_2=-c_2x_0-c_1x_1-3x_0^2x_1. \end{aligned}\right\} \qquad (6.14)$$

The structure of the above equations suggests further procedure: the first equation can be solved for x_0, then x_1 can be determined from the second equation, and x_2 from the third equation.

Assume the initial conditions in the following form: $x=a$ and $v=0$ at $t=0$. Based on expression (6.12), we obtain

$$x_0(0)+\alpha x_1(0)+\alpha^2x_2(0)=a,$$
$$\dot{x}_0(0)+\alpha\dot{x}_1(0)+\alpha^2\dot{x}_2(0)=0.$$

In order for these equalities to be satisfied for any value of α, the following six conditions must be fulfilled simultaneously

$$\left.\begin{aligned} &x_0(0)=a, \qquad &&\dot{x}_0(0)=0, \\ &x_1(0)=0, &&\dot{x}_1(0)=0, \\ &x_2(0)=0, &&\dot{x}_2(0)=0. \end{aligned}\right\} \qquad (6.15)$$

Solving the first equation of system (6.14) with the initial conditions indicated by the first line of system (6.15), we find

$$x_0=a\cos pt.$$

Substituting this expression in the second equation of system (6.14), we obtain

$$\ddot{x}_1 + p^2 x_1 = -c_1 a \cos pt - a^3 \cos^3 pt =$$
$$= -\left(c_1 a + \frac{3}{4} a^3\right) \cos pt - \frac{1}{4} a^3 \cos 3pt. \tag{6.16}$$

Suppose that the coefficient of cos pt is different from zero; then the solution of this equation will involve a so-called secular term of the form $t \sin pt$ in which the time t appears outside the sign of trigonometric functions. This solution of the resonant type can be used only for very small values of t since the secular term increases without limit as the argument t increases. For the solution to be valid for all values of t, it is necessary to eliminate the secular term from expression (6.16) putting

$$c_1 a + \frac{3}{4} a^3 = 0.$$

From this we determine the coefficient

$$c_1 = -\frac{3}{4} a^2. \tag{6.17}$$

The solution of Eq. (6.16) is then

$$x_1 = C_1 \cos pt + C_2 \sin pt + \frac{a^3}{32p^2} \cos 3pt.$$

After determining the constants from the second line of the initial conditions (6.15) we find

$$x_1 = \frac{a^3}{32p^2} (\cos 3pt - \cos pt).$$

Thus, the first two terms in expansions (6.12) and (6.13) are determined, giving, to the first order in α,

$$x = a \cos pt + \frac{\alpha a^3}{32p^2} (\cos 3pt - \cos pt)$$

and, in accordance with formulas (6.13) and (6.17),

$$p^2 = p_0^2 + \frac{3}{4} \alpha a^2. \tag{6.18}$$

Substitute now x_0 and x_1 in the third equation of (6.14) and solve it in a similar manner. This will give the third term of the expansion, the function x_2, with the quantity c_2 again determined so as to eliminate the secular term. Omitting the computations, we have, to the second order in α,

$$x = a\cos pt + \frac{\alpha a^3}{32p^2}(\cos 3pt - \cos pt) + \\ + \frac{\alpha^2 a^5}{1{,}024p^4}(\cos 5pt + 3\cos 3pt - 4\cos pt); \qquad (6.19)$$

$$p^2 = p_0^2 + \frac{3}{4}\alpha a^2 + \frac{3\alpha^2 a^4}{128p^2}.$$

In cases of other non-linear characteristics the procedure is analogous but the computations are generally more involved. Clearly, the method is adapted only to characteristics defined by a single analytic expression and cannot be used directly in other cases (for instance, for broken characteristics).

Note the important features of the solutions obtained: the vibratory process is described not by a single harmonic (a cosine curve) but by a sum of harmonics, the successive harmonics having relatively small amplitudes and hence being less significant; of course, the frequency of the fundamental harmonic p depends on the amplitude of vibration a.

Bubnov-Galerkin method. The application of the method will be illustrated by considering the problem of free vibration of a non-linear system when the equation is given by (6.2).

We shall seek the solution in the form

$$x = a\cos(pt + \alpha), \qquad (6.20)$$

where a, α, p are constants.

Substitute this solution in Eq. (6.2); of course, the result is not identically zero since expression (6.20) is not the exact solution of Eq. (6.2). According to the basic idea of the method, we require that the following integral taken between the limits of one period be zero

$$\int_0^{\frac{2\pi}{p}} [\ddot{x} + f(x)]\, x\, dt = 0. \qquad (6.21)$$

Substituting expression (6.20), we obtain

$$\int_0^{\frac{2\pi}{p}} \{-ap^2\cos(pt+\alpha) f[a\cos(pt+\alpha)]\}\cos(pt+\alpha)\, dt = 0 \qquad (6.22)$$

or

$$-\pi pa + \int_0^{\frac{2\pi}{p}} f[a\cos(pt+\alpha)]\cos(pt+\alpha)\, dt = 0.$$

Putting now $pt+\alpha=\psi$, we obtain the final formula for the square of the frequency

$$p^2=\frac{1}{\pi a}\int_0^{2\pi} f(a\cos\psi)\cos\psi\,d\psi. \tag{6.23}$$

We shall illustrate the application of this formula by taking the previously considered cubic characteristic

$$f(x)=p_0^2x+\alpha x^3.$$

In this case

$$f(a\cos\psi)=ap_0^2\cos\psi+\alpha a^3\cos^3\psi$$

and, by formula (6.23), we have

$$p^2=\frac{1}{\pi a}\int_0^{2\pi}(ap_0^2\cos\psi+\alpha a^3\cos^3\psi)\cos\psi\,d\psi.$$

Since

$$\int_0^{2\pi}\cos^2\psi\,d\psi=\pi,\qquad \int_0^{2\pi}\cos^4\psi\,d\psi=\frac{3}{4}\pi,$$

then

$$p^2=p_0^2+\frac{3}{4}\alpha a^2.$$

This result is identical with formula (6.18) derived by the perturbation method.

In the foregoing examples of application of the Bubnov-Galerkin method, the motion is defined by expression (6.20); the higher harmonics are eliminated from the start. However, the Bubnov-Galerkin method permits higher approximations to be constructed as well. For this purpose, it is necessary to seek the solution not in the form of the single function (6.20) but in the form of the functional series

$$x=a_1x_1+a_2x_2+\ldots$$

assigning the functions $x_1(t)$, $x_2(t)$, ... and then requiring that the integrals

$$\int_0^T[\ddot{x}+f(x)]\,x_i\,dt=0\qquad i=(1,2,\ldots)$$

be zero.

Harmonic linearization method (the simplest variant of the Krylov-Bogolyubov method).

Represent Eq. (6.2) as

$$\ddot{x}+p^2x=p^2x-f(x),$$

where p is a frequency of vibration not yet known.

Substituting on the right-hand side the approximate solution

$$x=a\cos(pt+\alpha),$$

we obtain the equation of forced vibration of a linear system

$$\ddot{x}+p^2x=F(t), \tag{6.24}$$

where

$$F(t)=ap^2\cos(pt+\alpha)-f[a\cos(pt+\alpha)]$$

is a periodic function with period $2\pi/p$.

Expanding $F(t)$ in a Fourier series, we have

$$F(t)=a_0+a_1\cos(pt+\alpha)+a_2\cos 2(pt+\alpha)+\dots$$

If the coefficient a_1 is different from zero, the term $a_1\cos(pt+\alpha)$ will bring about the *secular term* in solution of Eq. (6.24). To eliminate the secular term, it is necessary to put the Fourier coefficient a_1 equal to zero

$$a_1=\frac{2}{T}\int_0^T F(t)\cos(pt+\alpha)\,dt=0,$$

i.e.,

$$\int_0^{\frac{2\pi}{p}}\{ap^2\cos(pt+\alpha)-f[a\cos(pt+\alpha)]\}\cos(pt+\alpha)\,dt=0.$$

The last relation is identical with the basic Bubnov-Galerkin Equation (6.22)

Direct linearization method (the case of a symmetric characteristic). The method is based on the replacement of the non-linear characteristic $f(x)$ by the linear expression

$$f_*(x)=p^2x \tag{6.25}$$

with a suitably chosen coefficient p^2. The deviation of the approximating characteristic (6.25) from the approximated characteristic $f(x)$ is a function of the co-ordinate x (see Fig. 42)

$$r(x)=f(x)-p^2x$$

and may be required to make the integral

$$I = \int_0^a r^2\,dx$$

minimum. This integral represents the integral square deviation $r(x)$ over the entire range of the co-ordinate x. Obviously, it depends on the choice of the parameter p^2; hence the minimization is accomplished by determining this parameter from the equation $dI/d(p^2) = 0$.

With this approach, equal deviations r are assumed to be equally important regardless of the value of the co-ordinate x. Actually, however, in vibration problems the deviations r for larger values of the co-ordinate x are obviously more significant; hence it is more natural to consider the "weighed" deviation

$$rx = [f(x) - p^2 x]\,x.$$

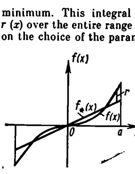


Fig. 42

The problem is then reduced to that of minimizing the integral

$$I_1 = \int_{-a}^{a} \{[f(x) - p^2 x]\,x\}^2\,dx,$$

i.e., determining p^2 from the equation

$$\frac{dI_1}{d(p_*^2)} = 0. \qquad (6.26)$$

This approach presupposes that the importance (weight) of the deviation is proportional to the corresponding value of the co-ordinate. From Eq. (6.26) we find

$$p^2 = \frac{5}{2a^5}\int_{-a}^{a} f(x)\,x^3\,dx = \frac{5}{a^5}\int_0^a f(x)\,x^3\,dx. \qquad (6.27)$$

After the parameter p^2 is found, the problem is reduced to the standard linear equation

$$\ddot{x} + p^2 x = 0$$

which replaces the given non-linear equation. From this it is clear that the parameter p^2 is the square of the frequency of free vibration.

As an example let us determine the frequency p by formula (6.27) for the characteristic considered above

$$f(x) = p_0^2 x + \alpha x^3.$$

We calculate

$$\int_0^a (p_0x+\alpha x^3)\,x^3\,dx=\frac{p_0^2a^5}{5}+\frac{\alpha a^7}{7}.$$

By formula (6.27) we obtain

$$p^2=p_0^2+\frac{5}{7}\alpha a^2. \tag{6.28}$$

Let us compare the accuracy of the results obtained by different methods in the particular case when $p_0=0$, i.e., $f(x)=\alpha x^3$.

By formula (6.28)

$$p=0.845a\sqrt{\alpha}.$$

By the formulas of the perturbation method (6.18) and of the Bubnov-Galerkin method (6.23), to a first approximation,

$$p=0.866a\sqrt{\alpha}.$$

By the exact formula (6.7)

$$p=0.847a\sqrt{\alpha}.$$

Let us plot relations $a(p^2)$ for systems containing non-linear elastic couplings with an initial compression or a clearance (see Fig. 38*e* and *f*). For this purpose, we use formula (6.27) which in the case of torsional vibration becomes

$$p^2=\frac{5}{a^5}\int_0^a f(\varphi)\,\varphi^3\,d\varphi, \tag{6.29}$$

where a is the amplitude of the angle of rotation.

For the coupling with an initial compression (see Fig. 38*e*), the calculation by formula (6.29) yields

$$p^2=\frac{knr^2}{J}\left(1+\frac{5P_0}{4kra}\right), \tag{6.30}$$

where J is the moment of inertia of the system. The curve (Fig. 43*a*) represents the relation $a(p^2)$ corresponding to formula (6.30). The dashed line gives the same relation for a larger value of the force P_0. The expression $\sqrt{knr^2/J}$ is the natural frequency when $P_0=0$ (of the linear system).

For the coupling with clearances (see. Fig. 38*f*), according to formula (6.29),

$$p^2=\frac{knr^2}{J}\left[1+\frac{1}{4}\left(\frac{\varphi_0}{a}\right)^5-\frac{5}{4}\frac{\varphi_0}{a}\right].$$

The curve $a(p^2)$ is shown in Fig. 43b. The dashed line corresponds to a larger value of φ_0. The expression $\sqrt{knr^2/J}$ is the natural frequency of the corresponding linear system ($\varphi_0 = 0$).

Direct linearization method (the case of an unsymmetric characteristic). If the characteristic is not symmetric (Fig. 44) and the initial

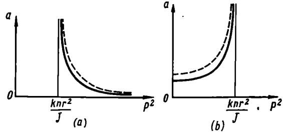


Fig. 43

deflection is a_1, the maximum deflection to the other side will be a_2, with $a_1 \neq a_2$ in the general case. The relation between these maximum deflections is given by the formula

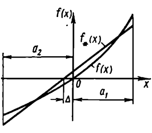


Fig. 44

$$\int_{-a_2}^{a_1} f(x)\,dx = 0$$

which expresses the equality of the potential energies of the system at the two extreme positions. The central position about which the system is vibrating is shifted to the left of the origin by the distance

$$\Delta = \frac{a_2 - a_1}{2}.$$

To any given deflection a_1 correspond certain distinct values of the deflection a_2 and the shift of the centre of oscillation Δ. The approximating linear characteristic should be drawn through the centre of vibration; its equation is

$$f_*(x) = p^2(x + \Delta).$$

We form, as before, the deviation

$$r = f(x) - p^2(x + \Delta),$$

and take the co-ordinate $x+\Delta$ as the "weight". It is then necessary to minimize the integral

$$I=\int_{-a_2}^{a_1}\{[f(x)-p^2(x+\Delta)](x+\Delta)\}^2\,dx.$$

Setting up the equation

$$\frac{dI}{d(p^2)}=0,$$

we obtain

$$p^2=\frac{5}{(a_1+a_2)^5}\int_{-a_2}^{a_1}f(x)(x+\Delta)^3\,dx.$$

Introduce the variable

$$x_1=x+\Delta$$

and half the total excursion (or the amplitude of vibration in an equivalent linear system)

$$a=\frac{a_1+a_2}{2}.$$

Then

$$p^2=\frac{5}{2a^5}\int_{-a}^{a}f(x_1-\Delta)\,x_1^3\,dx_1 \tag{6.31}$$

is the formula for the square of the frequency of free vibration.

7. Linear Systems of Several Degrees of Freedom

The Simplest System

Derivation of Equations of Problem. As stated above, the equations of motion can be set up by using the fundamental method (Lagrange's equations), the direct and inverse methods. Their application will be illustrated by considering the simplest example of a *two-mass system* (Fig. 45*a*) in which k_1 and k_2 are the stiffnesses of the springs, m_1 and m_2 the masses, x_1 and x_2 the displacements of the masses.

Fundamental method. During vibration forces $N_1=k_1x_1$ and $N_2=k_2(x_2-x_1)$ develop in the springs, therefore the potential energy of the system is expressed as

$$U=\frac{k_1x_1^2}{2}+\frac{k_2(x_2-x_1)^2}{2}.$$

The kinetic energy of the system is

$$T = \frac{m_1\dot{x}_1^2}{2} + \frac{m_2\dot{x}_2^2}{2}.$$

These expressions must be substituted in Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} = -\frac{\partial U}{\partial x_i} \quad (i = 1, 2). \tag{7.1}$$

We first form the corresponding derivatives

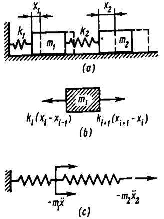


Fig. 45

$$\frac{\partial T}{\partial \dot{x}_1} = m_1\dot{x}_1, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1,$$

$$\frac{\partial T}{\partial \dot{x}_2} = m_2\dot{x}_2, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2,$$

$$\frac{\partial U}{\partial x_1} = k_1x_1 - k_2(x_2 - x_1),$$

$$\frac{\partial U}{\partial x_2} = k_2(x_2 - x_1);$$

accordingly Eqs. (7.1) become

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) &= 0, \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) &= 0. \end{aligned}\right\} \tag{7.2}$$

Direct method. During vibration the first mass is acted on by the force in the first spring and the force in the second spring, their projections on the x axis being equal to $-k_1x_1$ and $k_2(x_2 - x_1)$, respectively; the equation of motion of the first mass is

$$-k_1x_1 + k_2(x_2 - x_1) = m_1\ddot{x}_1. \tag{7.3}$$

The second mass is acted on by the force in the second spring only, giving a projection $-k_2(x_2 - x_1)$; the equation of motion is

$$-k_2(x_2 - x_1) = m_2\ddot{x}_2. \tag{7.4}$$

With a large number of series-connected masses, each of the equations will involve *no more than three unknown co-ordinates* as the spring forces acting on the ith mass are completely determined by the displacements x_{i-1}, x_i and x_{i+1} (Fig. 45*b*).

Inverse method. Consider a massless system consisting of two springs subjected to inertia forces (Fig. 45*c*). The first spring is subjected to a force $-m_1\ddot{x}_1 - m_2\ddot{x}_2$ and the second to a force $-m\ddot{x}_2$. The displacement of the first point (equal to the extension of the first spring) is

$$x_1 = \frac{-m_1\ddot{x}_1 - m_2\ddot{x}_2}{k_1} \tag{7.5}$$

and the displacement of the second point (equal to the total extension of the two springs) is

$$x_2 = \frac{-m_1\ddot{x}_1 - m_2\ddot{x}_2}{k_1} + \frac{-m_2\ddot{x}_2}{k_2} \tag{7.6}$$

Comparing the above equations, we note the following.

The identity of Eqs. (7.2) derived by the fundamental method and Eqs. (7.3), (7.4) derived by the direct method is not coincidental. It results whenever the generalized co-ordinates are chosen so that the expression for the kinetic energy does not contain products of the velocities (the product $\dot{x}_1\dot{x}_2$ in the present case). If the generalized co-ordinates are chosen so that the expression for the potential energy will not contain the product x_1x_2 (this was not accomplished in our problem), the equations derived by the fundamental method will be identical with the equations set up by using the inverse method.

Although the system of equations (7.3) and (7.4) is, of course, equivalent to the system of equations (7.5) and (7.6), they are not identical as to the amount of numerical work involved when the number of masses is larger. As stated above, when applied to mechanical systems of the type considered, the direct method, regardless of the number of masses, will lead to equations each of which contains no more than three unknown functions. At the same time the equations set up by the inverse method will become progressively more complicated as each of them will include all the n unknown functions of the problem.

For this reason the direct method is always preferred for the analysis of vibrations of chain systems of the type considered. In other cases it is more convenient to apply the inverse method (as in the analysis of flexural vibrations of beams with several masses).

Solution of equations of motion for the simplest system. We proceed with the consideration of the system of two degrees of freedom (Fig. 45*a*); this will provide a simple way of disclosing the basic features of vibrating systems having several degrees of freedom, particularly the existence of *several natural frequencies*. Let us try to satisfy the equations of vibration (7.3) and (7.4) by the functions

$$\left.\begin{aligned} x_1 &= a_1 \sin(pt + \alpha), \\ x_2 &= a_2 \sin(pt + \alpha). \end{aligned}\right\} \tag{7.7}$$

These functions do not yet represent the general solution of the problem but they offer a means of constructing it. Substituting expressions (7.7) in Eqs. (7.3) and (7.4), we obtain

$$\left.\begin{aligned} -k_1a_1+k_2(a_2-a_1)&=-m_1a_1p^2, \\ -k_2(a_2-a_1)&=-m_2a_2p^2. \end{aligned}\right\} \qquad (7.8)$$

These equations involve three unknowns—the amplitudes a_1, a_2 and the frequency p. Of course, it is impossible to find three quantities from two equations; however, Eqs. (7.8) allow one to carry through the determination of the natural frequency.

We find from the first equation the ratio

$$\frac{a_2}{a_1}=\frac{k_1+k_2-m_1p^2}{k_2} \qquad (7.9)$$

and the same ratio from the second equation

$$\frac{a_2}{a_1}=\frac{k_2}{k_2-m_2p^2}\,. \qquad (7.10)$$

The compatibility of expressions (7.9) and (7.10) leads to the *frequency equation*

$$\frac{k_1+k_2-m_2p^2}{k_2}=\frac{k_2}{k_2-m_2p^2} \qquad (7.11)$$

which contains a single unknown, the frequency p.

The frequency equation (7.11) can also be obtained in a different way. The system of equations (7.8) may be rewritten as

$$\begin{aligned} (k_1+k_2-m_1p^2)\,a_1-k_2a_2&=0, \\ -k_2a_1+(k_2-m_2p^2)\,a_2&=0. \end{aligned}$$

Here the *homogeneity* of the system of equations in the amplitudes a_1 and a_2 is clearly seen. The homogeneous system of equations is satisfied by the zero roots

$$a_1=a_2=0.$$

This trivial solution means the absence of vibrations in our problem. However, in one special case a non-trivial solution ($a_1\neq 0$, $a_2\neq 0$) is possible, namely, when the determinant of the coefficients of the system is equal to zero

$$\begin{vmatrix} k_1+k_2-m_1p^2 & -k_2 \\ -k_2 & k_2-m_2p^2 \end{vmatrix}=0. \qquad (7.11a)$$

Expanding the determinant we arrive at Eq. (7.11) or, carrying out further computations, at the equation

$$p^4-\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)p^2+\frac{k_1k_2}{m_1m_2}=0. \qquad (7.11b)$$

We obtain two real and positive solutions for the square of the frequency

$$\left.\begin{aligned} p_1^2&=\frac{1}{2}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)-\sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)^2-\frac{k_1k_2}{m_1m_2}}\,,\\ p_2^2&=\frac{1}{2}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)+\sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)^2-\frac{k_1k_2}{m_1m_2}}\,. \end{aligned}\right\} \tag{7.12}$$

Accordingly we can calculate two natural frequencies

$$\left.\begin{aligned} p_1&=\sqrt{\frac{1}{2}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)-\sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)^2-\frac{k_1k_2}{m_1m_2}}}\,,\\ p_2&=\sqrt{\frac{1}{2}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)+\sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1}+\frac{k_2}{m_2}\right)^2-\frac{k_1k_2}{m_1m_2}}}\,. \end{aligned}\right\} \tag{7.13}$$

Thus, the vibratory process is a *two-frequency* one and is determined by the functions $\sin(p_1t+\alpha_1)$ and $\sin(p_2t+\alpha_2)$. The existence of two frequencies indicates the incompleteness of the way of writing (7.7). In order to reflect both harmonics in the general solution, we introduce more complicated subscripts and write the solution as

$$\left.\begin{aligned} x_1&=a_{11}\sin(p_1t+\alpha_1)+a_{12}\sin(p_2t+\alpha_2),\\ x_2&=a_{21}\sin(p_1t+\alpha_1)+a_{22}\sin(p_2t+\alpha_2). \end{aligned}\right\} \tag{7.14}$$

Here the first subscript of the amplitude a_{ik} designates the number of the co-ordinate and the second subscript the number of the term in the line (the number of the frequency).

The amplitudes are related by expression (7.9) or the equivalent expression (7.10). Substituting the first frequency p_1 in expression (7.9), we obtain the amplitude ratio of the first harmonic, which is independent of the initial conditions

$$\varkappa_{21}=\frac{a_{21}}{a_{11}}=\frac{k_1+k_2-m_1p_1^2}{k_2}. \tag{7.15}$$

The relation between the amplitudes of the second harmonic is given by the same expression (7.9) or expression (7.10) if we put $p^2=p_2^2$ in it

$$\varkappa_{22}=\frac{a_{22}}{a_{12}}=\frac{k_1+k_2-m_1p_2^2}{k_2}. \tag{7.16}$$

Consequently, in place of solutions (7.14) we can write

$$\left.\begin{aligned} x_1&=a_{11}\sin(p_1t+\alpha_1)+a_{12}\sin(p_2t+\alpha_2),\\ x_2&=\varkappa_{21}a_{11}\sin(p_1t+\alpha_1)+\varkappa_{22}a_{12}\sin(p_2t+\alpha_2). \end{aligned}\right\} \tag{7.17}$$

Here the natural frequencies p_1 and p_2 and the ratios $\varkappa_{21}$ and $\varkappa_{22}$ depend only on the parameters of the vibrating system. The quantities a_{11}, a_{12}, α_1 and α_2 are to be determined from four initial conditions which express the values of displacements and velocities of both masses at the initial instant.

Let, for example, at $t = 0$

$$x_1(0) = 0, \qquad x_2(0) = 0,$$
$$\dot{x}_1(0) = 0, \qquad \dot{x}_2(0) = v_0,$$

i.e., the motion of the system be produced by an instantaneous impact on the second mass (this kind of problem occurs, for example, in the analysis of vibration of artillery systems after firing).

Using expressions (7.17) we obtain

$$a_{11}\sin\alpha_1 + a_{12}\sin\alpha_2 = 0,$$
$$\varkappa_{21}a_{11}\sin\alpha_1 + \varkappa_{22}a_{12}\sin\alpha_2 = 0,$$
$$a_{11}p_1\cos\alpha_1 + a_{12}p_2\cos\alpha_2 = 0,$$
$$\varkappa_{21}a_{11}p_1\cos\alpha_1 + \varkappa_{22}p_2\cos\alpha_2 = v_0,$$

whence

$$\alpha_1 = \alpha_2 = 0, \quad a_{11} = \frac{v_0}{p_1}\,\frac{1}{\varkappa_{21}-\varkappa_{22}}, \quad a_{12} = \frac{v_0}{p_2}\,\frac{1}{\varkappa_{22}-\varkappa_{21}}.$$

The quantities p_1, p_2, $\varkappa_{21}$ and $\varkappa_{22}$ are known; they are calculated with formulas (7.13), (7.15) and (7.16). Thus, *both vibratory components appearing in expression (7.17) are realized* in the case under consideration.

By suitably choosing the initial conditions it is possible to obtain a *monofrequent vibration*, for example, to have $a_{12} = 0$. The vibration is then defined by a single harmonic

$$x_1 = a_{11}\sin(p_1t + \alpha_1),$$
$$x_2 = \varkappa_{21}a_{11}\sin(p_1t + \alpha_1).$$

Since the coefficient $\varkappa_{21}$ is independent of the initial conditions, this monofrequent vibration is characterized by a definite amplitude ratio which depends only on the parameters of the system and remains unchanged during vibration. This ratio defines *the first natural mode of vibration.*

If the initial conditions are such that $a_{11} = 0$, the vibration will also be monofrequent but with the frequency p_2

$$x_1 = a_{12}\sin(p_2t + \alpha_2),$$
$$x_2 = \varkappa_{22}a_{12}\sin(p_2t + \alpha_2).$$

Theamplitude ratio $\varkappa_{22}$ defines *the second natural mode of vibration.*

In the special case when $k_1 = k_2 = k$, $m_1 = m_2 = m$, we obtain from formulas (7.12)

$$p_1^2 = \frac{k}{m}\,\frac{3-\sqrt{5}}{2}, \quad p_2^2 = \frac{k}{m}\,\frac{3+\sqrt{5}}{2}$$

and from formulas (7.15) and (7.16)

$$\varkappa_{21} = \frac{1+\sqrt{5}}{2} = 1.618, \quad \varkappa_{22} = \frac{1-\sqrt{5}}{2} = -0.618.$$

In the first natural mode of vibration both masses move in the same direction, the amplitude of vibration of the second mass being larger than that of the first mass. To the second natural mode of vibration corresponds the motion of the masses in opposite directions; the amplitude of vibration of the second mass is smaller than that of the first mass.

We emphasize again that the realization of natural modes in pure form requires a suitable choice of initial conditions.

Orthogonality of natural modes of vibration. When the system is vibrating in the first natural mode the maximum deflections are a_{11} and a_{21}; to these deflections correspond inertia forces $m_1a_{11}p_1^2$ and $m_2a_{21}p_1^2$. Likewise, when the system is vibrating in the second natural mode the maximum deflections are a_{12} and a_{22} and the corresponding inertia forces are equal to $m_1a_{12}p_2^2$ and $m_2a_{22}p_2^2$.

Let us apply Betti's theorem of reciprocity of virtual work to these two states. According to this theorem, the work done by the forces of the first state ($m_1a_{11}p_1^2$, $m_2a_{21}p_1^2$) during the displacements of the second state (a_{12}, a_{22}) is equal to the work done by the forces of the second state ($m_1a_{12}p_2^2$, $m_2a_{22}p_2^2$) during the displacements of the first state (a_{11}, a_{21}), i.e.,

$$p_1^2(m_1a_{11}a_{12} + m_2a_{21}a_{22}) = p_2^2(m_1a_{12}a_{11} + m_2a_{22}a_{21})$$

or

$$(p_2^2 - p_1^2)(m_1a_{11}a_{12} + m_2a_{21}a_{22}) = 0.$$

Since p_1^2 and p_2^2 are different, the following equality must be fulfilled

$$m_1a_{11}a_{12} + m_2a_{21}a_{22} = 0.$$

This relation expresses the *orthogonality property* of two natural modes of vibration; after dividing by $a_{11}a_{12}$ it can also be rewritten as

$$m_1 + m_2\varkappa_{21}\varkappa_{22} = 0.$$

If the ratio $\varkappa_{21}$ defining the first natural mode is known, the ratio $\varkappa_{22}$ corresponding to the second natural mode can be found from the orthogonality condition

$$\varkappa_{22} = -\frac{m_1}{m_2\varkappa_{21}}\,.$$

The concept of natural modes of vibration, as well as the important property of orthogonality, will be used later in dealing with systems having an arbitrary finite number of degrees of freedom. *The number of natural modes of vibration and also the number of natural frequencies are each equal to the number of degrees of freedom of the system.*

Torsional Vibrations of Shafts

Basic equations. Consider torsional vibrations of a multimass system (Fig. 46) which is a conventional equivalent schematic diagram (though not without defect) for the analysis of torsional vibrations of crankshafts. A crankshaft is reduced to the equivalent scheme by making the following substitutions:

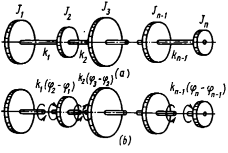

J_1
J_2
J_3
J_{n-1}
J_n
k_1
k_2
k_{n-1}
(a)
$k_1(\varphi_2-\varphi_1)$
$k_2(\varphi_3-\varphi_2)$
$k_{n-1}(\varphi_n-\varphi_{n-1})$
(b)

Fig. 46

the moment of inertia of the substitution disk about the shaft axis must be equal to the moment of inertia of the crank about the same axis; the additional effective mass of the connecting rod is taken into account;

the torsional stiffness of a portion of the substitution shaft must be equal to the torsional stiffness of the corresponding portion of the crankshaft.

It should be remembered that these substitutions cannot ensure complete equivalence of the two schemes. The point is that the reduced mass moment of inertia of the crank and connecting rod varies during rotation of the crankshaft; therefore the replacement of the crank by a disk of constant moment of inertia is not strictly justifiable. Moreover, when the crankshaft is subjected to two opposite couples (Fig. 47) the deformation consists not only in twisting the portion between the couples; the other portions will also be twisted due to bending. The substitution system is not capable of reflecting this side effect.

Nevertheless, experiments confirm the acceptability of the substitution scheme if equivalent moments of inertia and particularly equivalent stiffnesses are determined with sufficient accuracy.

Denote (see Fig. 46*a*) by J_1, $J_2, \ldots, J_n$ the mass moments of inertia of the disks about the shaft axis, by $k_1, k_2, \ldots, k_{n-1}$ the torsional stiffnesses of the portions of the shaft, by $\varphi_1, \varphi_2, \ldots, \varphi_n$ the angles of rotation of the disks about the shaft axis.

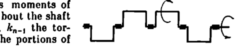


Fig. 47

The twisting moments acting at the sections of the shaft depend on the relative rotation of two adjacent disks and are given by the following formulas:

in the first portion

$$k_1(\varphi_2-\varphi_1),$$

in the second portion

$$k_2(\varphi_3-\varphi_2),$$

$$\cdots\cdots$$

in the $(n-1)$th portion

$$k_{n-1}(\varphi_n-\varphi_{n-1}).$$

The equations of motion are most easily derived by the direct method (see Fig. 46*b*)

$$\left.\begin{aligned}
k_1(\varphi_2-\varphi_1)&=J_1\ddot{\varphi}_1,\\
k_2(\varphi_3-\varphi_2)-k_1(\varphi_2-\varphi_1)&=J_2\ddot{\varphi}_2,\\
k_3(\varphi_4-\varphi_3)-k_2(\varphi_3-\varphi_2)&=J_3\ddot{\varphi}_3,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots\\
k_{n-1}(\varphi_n-\varphi_{n-1})-k_{n-2}(\varphi_{n-1}-\varphi_{n-2})&=J_{n-1}\ddot{\varphi}_{n-1},\\
-k_{n-1}(\varphi_n-\varphi_{n-1})&=J_n\ddot{\varphi}_n.
\end{aligned}\right\} \tag{7.18}$$

The number of these equations is equal to the number of disks.

Equations (7.18) are satisfied by the solution

$$\varphi_1=\varphi_2=\ldots=\varphi_n=a_0+\omega t \tag{7.19}$$

which defines a *uniform rotation* of the shaft and disks as a rigid body. Besides, the following solution is possible

$$\left.\begin{aligned}\varphi_1 &= a_1 \sin(pt+\alpha),\\ \varphi_2 &= a_2 \sin(pt+\alpha),\\ &\dots\dots\dots\dots\\ \varphi_n &= a_n \sin(pt+\alpha),\end{aligned}\right\} \tag{7.20}$$

which defines *elastic vibrations*.

Check that expressions (7.20) do actually represent a solution of the system of equations (7.18) subject to certain relations between the amplitudes $a_1, a_2, \ldots, a_n$. Substituting expressions (7.20) in Eqs. (7.18), we obtain

$$\left.\begin{aligned}k_1(a_2-a_1) &= -J_1p^2a_1,\\ k_2(a_3-a_2)-k_1(a_2-a_1) &= -J_2p^2a_2,\\ k_3(a_4-a_3)-k_2(a_3-a_2) &= -J_3p^2a_3,\\ \dots\dots\dots\dots\dots\dots\dots\dots&\dots\dots\dots\\ k_{n-1}(a_n-a_{n-1})-k_{n-2}(a_{n-1}-a_{n-2}) &= -J_{n-1}p^2a_{n-1},\\ -k_{n-1}(a_n-a_{n-1}) &= -J_np^2a_n.\end{aligned}\right\} \tag{7.21}$$

This system of equations involves $n+1$ unknowns (n unknown amplitudes and the frequency p).

Frequency equation. Based on system (7.21), it is possible to obtain an equation containing one unknown, p^2, and defining the frequency spectrum $p_1, p_2, \ldots$, of the multimass system under consideration. To do this, we express a_2 in terms of a_1 from the first equation of system (7.21) and substitute the result in the second equation of the system; we can now express a_3 in terms of a_1. Carrying on this process to the $(n-1)$th equation, we express all the amplitudes (to the nth inclusive) in terms of the amplitude a_1. These expressions will depend on the frequency p, the dependence becoming progressively complicated.

Finally, substituting the expressions for a_{n-1} and a_n in the last equation of (7.21), we can eliminate a_1 and obtain an algebraic equation containing one unknown, the frequency p; the degree of this equation with respect to the square of the frequency p^2 is equal to the number of disks n.

The frequency equation can be derived in a different way, if system (7.21) is treated as a system of linear algebraic equations for the unknown amplitudes $a_1, a_2, \ldots, a_n$. Since this system is homogeneous, non-zero solutions are possible only if the determinant of the coefficients of the amplitudes is equal to zero.

For instance, with three disks ($n = 3$) this condition is

$$\begin{vmatrix} -k_1+J_1p^2 & k_1 & 0 \\ k_1 & -k_2-k_1+J_2p^2 & k_2 \\ 0 & k_2 & -k_2+J_3p^2 \end{vmatrix} = 0. \qquad (7.22)$$

Expanding the determinant, we obtain the *frequency equation*. An inspection of the structure of the principal diagonal of the determinant indicates that the degree of this equation with respect to p^2 is n.

For a three-mass system, expanding determinant (7.22) gives

$$p^2\left[\frac{J_1J_2J_3}{k_1k_2}p^4-\left(\frac{J_2+J_3}{k_1}J_1+\frac{J_1+J_2}{k_2}J_3\right)p^2+J_1+J_2+J_3\right]=0. \qquad (7.23)$$

Corresponding to the degree of the equation the number of roots p_i is also n; one of the roots *is always zero* so that the number of non-zero frequencies is one unit less than the number of disks and is equal to $n-1$. The zero root corresponds to a rotation of all disks and shaft as a rigid body; the non-zero roots correspond to the phenomenon of elastic vibration. Consequently, a system consisting of a weightless shaft and n disks possesses $n-1$ non-zero natural frequencies $p_1, p_2, \ldots$; they are usually arranged in order of ascending values of the frequency.

Example 8. Determine the natural frequencies for a three-mass system (Fig. 48*a*) which represents a simplified scheme of a ship diesel engine. The reduced moment of inertia of the engine shaft is $J_1 = 17{,}000$ kgf-cm-sec^3, the moment of inertia of the flywheel $J_2 = 85{,}000$ kgf-cm-sec^2; the reduced moment of inertia of the propeller (with the mass of the propeller shaft and the added mass of water taken into account) $J_3 = 27{,}000$ kgf-cm-sec^2. The stiffnesses are: $k_1 = 211 \times 10^6$ kgf-cm, $k_2 = 14.8 \times 10^6$ kgf-cm.

With these values the frequency equation (7.23) becomes

$$p^2(0.01242p^4-195.3p^2+129{,}000)=0.$$

The non-zero roots of this equation are

$$p_1^2=690 \text{ sec}^{-2}; \qquad p_2^2=15{,}000 \text{ sec}^{-2}.$$

Consequently, the natural frequencies are $p_1 = 26.3$ sec^{-1}, $p_2 = 122.5$ sec^{-1}.

With four and more disks we have to deal with frequency equations of degree greater than two. The solution of such equations is cumbersome in the general case; however, the chain structure of Eqs. (7.21) affords a considerable simplification in determining frequencies by the method of successive approximations described below.

Natural modes of vibration. From the basic system of equations (7.21) it is possible to find not only the natural frequencies but also the corresponding ratios between the amplitudes. If the frequency p_1 found from the frequency equation is substituted in system (7.21), all the amplitudes can be expressed successively in terms of the amplitude a_1. If the frequency p_2 is substituted in system (7.21), some other relations between the amplitudes will result. Thus, to each frequency p_i corresponds its own set of amplitude ratios, i.e., *its own mode of vibration*; each of these is a natural mode of vibration. The number of these modes is equal to the number of natural frequencies, i.e., one unit less than the number of disks. An important property is the mutual orthogonality of any two natural modes

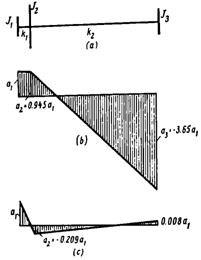


Fig. 48

$$J_1a_{1i}a_{1k}+J_2a_{2i}a_{2k}+\ldots+J_na_{ni}a_{nk}=0. \tag{7.24}$$

Example 9. Find the natural modes of vibration for the system represented in Fig. 48.

Since the system contains three disks, the number of these modes is two.

1. *Determination of first natural mode.* Substitute in Eqs. (7.21)

$$p^2=p_1^2=690\ \text{sec}^{-2}.$$

We find from the first equation

$$a_{21}=0.945a_{11}$$

and from the second equation

$$a_{31} = -3.65a_{11}$$

(the second subscripts designate the number of the mode).

The last result can also be obtained from the third equation, which is a consequence of the first two. Figure 48*b* shows the first natural mode having one node. As can be seen, the vibrations corresponding to the first frequency occur almost without deformations in the engine shaft and consist primarily in twisting the propeller shaft.

2. *Determination of second natural mode.* If the same operations are repeated for $p^2 = 15{,}000\ \text{sec}^{-2}$, we obtain $a_{22} = -0.209a_{12}$ and $a_{32} = 0.008a_{12}$. In this case the prime effect is the twisting of the engine shaft while the propeller shaft is only slightly twisted (Fig. 48*c*).

To the second frequency corresponds a two-node natural mode of vibration. It should be noted that for higher frequencies natural modes of vibration of multimass systems become increasingly complicated (as regards the number of nodes).

It may be checked that the modes found above are orthogonal to each other. Using formula (7.24) we calculate

$$J_1a_{11}a_{12} + J_2a_{21}a_{22} = J_3a_{31}a_{32} = 17{,}000a_{11}a_{12} +$$
$$+ 85{,}000 \times 0.945a_{11}(-0.209a_{12}) + 27{,}000(-3.65a_{11})\,0.008a_{12} \cong 0.$$

Besides, each one of these modes is orthogonal to the "zero" natural mode (rotation of the shaft as a rigid body); to the latter correspond equal displacements $a_{10} = a_{20} = a_{30} = a_0$. The orthogonality of the first and zero modes is checked by the calculation

$$J_1a_{11}a_{10} + J_2a_{21}a_{20} + J_3a_{31}a_{30} = 17{,}000a_{11}a_0 + 85{,}000 \times 0.945a_{11}a_0 +$$
$$+ 27{,}000(-3.65a_{11})\,a_0 \approx 0$$

and the orthogonality of the second and zero modes by the calculation

$$J_1a_{12}a_{10} + J_2a_{22}a_{20} + J_3a_{32}a_{30} = 17{,}000a_{12}a_0 + 85{,}000(-0.209a_{12})\,a_0 +$$
$$+ 27{,}000 \times 0.008a_{12}a_0 \approx 0.$$

Determination of motion from initial conditions. In writing solution (7.20) it was assumed that the vibrations were monofrequent, i.e., for any disk they were defined by one harmonic

$$\varphi_i = a_i \sin(pt + \alpha).$$

The existence of a series of values of the frequency p requires generalizing solution (7.20) and writing it as

$$\varphi_i = a_{i1}\sin(p_1t + \alpha_1) + a_{i2}\sin(p_2t + \alpha_2) + \dots$$
$$\dots + a_{i,\,n-1}\sin(p_{n-1}t + \alpha_{n-1}) \qquad (i = 1, 2, \dots, n).$$

Here the first subscript of the amplitude designates the number of the disk and the second, the number of the corresponding frequency.

To obtain the general solution, it is necessary to take into account a possible rotation of the whole system as a rigid body (this corresponds to the frequency $p = 0$), i.e., to add a term of the form given by (7.19)

$$\varphi_i = a_0 + \omega t + a_{i1}\sin(p_1 t + \alpha_1) + a_{i2}\sin(p_2 t + \alpha_2) + \cdots$$
$$\cdots + a_{i,\,n-1}\sin(p_{n-1} t + \alpha_{n-1}) \qquad (i = 1, 2, \ldots, n). \qquad (7.25)$$

Equations (7.25) involve $2n$ unknowns: $n-1$ unknown amplitudes of vibration of the first disk ($a_{11}, a_{12}, \ldots, a_{1,\,n-1}$), $n-1$ unknown initial phases ($\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$), the angular displacement a_0 and the angular velocity ω (the amplitudes of vibration of all the other disks a_{ik} are determined in terms of the amplitudes of the component vibrations of the first disk a_{1k}; the ratios a_{ik}/a_{1k} depend on the number of the frequency k and define the corresponding modes of vibration).

Thus, for the complete solution of the problem it is necessary and sufficient to specify $2n$ initial conditions, viz. the angular displacements and angular velocities of all the n disks.

With arbitrarily assigned initial conditions, the vibration of each disk will be multifrequency, i.e., will be made up of the sum of harmonics. If the initial angular displacements correspond to one of the natural modes of vibration, this and only this mode (and the corresponding frequency) will be realized in the further process; in the general case, however, the vibration is of a complex nature and represents a combination of n modes of vibration. The relative contribution of each one depends on how close the assigned system of initial displacements is to a particular natural mode.

Calculation of natural frequencies and modes by the method of successive approximations. The method of successive approximations (*method of residuals*) is efficient in the analysis of multimass systems ($n > 3$) for which even the first step of the exact solution (the expanding of the determinant and the derivation of the frequency equation) presents difficulties. The method is based on the utilization of the chain structure of system (7.21).

Assuming $a_1 = 1$ and assigning an arbitrary value of p^2, find the amplitude a_2 from the first equation of system (7.21), the amplitude a_3 from the second equation of the same system, the amplitude a_4 from the third equation and, finally, the amplitude a_n from the last but one equation. If the calculated values of a_{n-1} and a_n are substituted in the last equation

$$-k_{n-1}(a_n - a_{n-1}) + J_n p^2 a_n = 0$$

it will not in general be satisfied because of the arbitrary choice of p^2. The resulting value on the left-hand side (residual) characterizes the degree of inaccuracy of the assumed value of p^2 and at the same time indicates the way in which the computing value of p^2 should be changed in the next approximation.

Further the calculation is repeated for a new value of p^2; the sign and magnitude of a new residual will serve to indicate the necessary correction in the value of p^2. The calculation is repeated until a satisfactory result is obtained in the last equation.

It is advisable to arrange the computations in compact tabular form based on relations of the type

$$k_i (a_{i+1} - a_i) = -J_1 p^2 a_1 - J_2 p^2 a_2 - \dots - J_i p^2 a_i \qquad (7.26)$$

which result from Eqs. (7.21) after the first i equations of the system are added up. Relation (7.26) expresses that the twisting moment at a section of the ith portion of the shaft (the left-hand side) is equal to the sum of the moments of the inertia forces of all the disks located to the left (the right-hand side).

Assigning a value of p^2 and assuming $a_1 = 1$, we find from relation (7.26) for $i = 1$

$$a_2 = -\frac{J_1 p^2}{k_1} + 1.$$

Further we have from the same relation for $i = 2$

$$a_3 = -\frac{J_1 p^2 + J_2 p^2 a_2}{k_2} + a_2.$$

The general formula is

$$a_{i+1} = -\frac{\sum\limits_{k=1}^{k=i} J_k p^2 a_k}{k_i} + a_i. \qquad (7.26a)$$

In this way the process is carried on up to the $(n-1)$th equation. After a_n is found therefrom, we can proceed to the last equation and calculate its left-hand side. The result must be zero since, if all equations of the type of (7.26) are added up, we shall have

$$\sum_{k=1}^{k=n} J_k p^2 a_k = 0.$$

Because of the inaccuracy of the assumed value of p^2 the result will be different from zero. The residual obtained expresses an unbalanced torque which must be zero if the value of p^2 is properly chosen.

After a few calculations of this type (for different values of p^2) we can plot a curve of the residual R as a function of p^2 (Fig. 49). The points of intersection of the curve and the axis of abscissas correspond to the true values of the frequencies.

The amount of numerical work can be reduced if approximate values of the frequencies are known; they are often estimated roughly by replacing a given system by a simplified three-mass system.

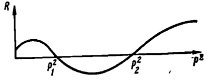


Fig. 49

Example 10. Consider a more accurate scheme of the ship diesel installation (Fig. 50*a*) treated above. The scheme consists of disks *1*, *2*, *3*, *4*, *5* and *6* to which are reduced the cranks of engines, flywheel *7* and propeller *8* with added masses of the propeller shaft and water.

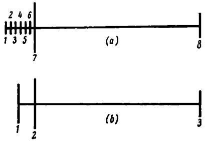


Fig. 50

The following data are given:

$$J_1 = J_2 = J_3 = J_4 = J_5 = J_6 = 2{,}830 \text{ kgf-cm-sec}^2,$$

$$J_7 = 85{,}000 \text{ kgf-cm-sec}^2, \quad J_8 = 27{,}000 \text{ kgf-cm-sec}^2,$$

$$k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 740 \times 10^6 \text{ kgf-cm}, \quad k_7 = 14.8 \times 10^6 \text{ kgf-cm}.$$

To determine approximately the two lowest frequencies, we form a simplified system (Fig. 50*b*); here the first six disks are replaced by a single one for which $J_1 = 6 \times 2{,}830 \approx 17{,}000$ kgf-cm-sec², and the numbers of the last two disks (flywheel and propeller) are changed.

The stiffness of the first portion in the simplified scheme is

$$k_1 = \frac{740 \times 10^6}{3.5} = 211.4 \times 10^6 \text{ kgf-cm}$$

(the length of portion *1-2* in the simplified scheme is 3.5 times the length of each of the portions between disks *1* to *6* in the given scheme). For this simplified scheme, $p_1^2 = 690$ sec^{-2} and $p_2^2 = 15{,}000$ sec^{-2}, as found in the examples above. We assume these values as starting ones and refine them by the *Holzer-Tolle method*.

Refinement of first frequency and first natural mode of vibration. Assuming $p_1^2 = 690$ sec^{-2}, we fill in Table 2.

Table 2

Computational Table for First Approximation ($p^2 = 690$ sec^{-2})

No. of disk	J	$Jp^2 \times 10^{-6}$	a	$Jp^2a \times 10^{-6}$	$\Sigma Jp^2a \times 10^{-6}$	$k \times 10^{-6}$	$\frac{\Sigma Jp^2a}{k}$
1	2,830	1.95	1.000	1.95	1.95	740	0.003
2	2,830	1.95	0.997	1.94	3.89	740	0.005
3	2,830	1.95	0.992	1.93	5.82	740	0.008
4	2,830	1.95	0.984	1.92	7.74	740	0.010
5	2,830	1.95	0.974	1.90	9.64	740	0.013
6	2,830	1.95	0.961	1.87	11.51	740	0.015
7	85,000	58.7	0.946	55.6	67.1	14.8	4.52
8	27,000	18.6	−3.57	−66.4	$\boxed{0.8}$		

The relatively small value of the residual (0.8×10^6) indicates that the assumed value $p^2 = 690$ sec^{-2} differs only slightly from the true one. As a second approximation we assume a somewhat different value, $p^2 = 700$ sec^{-2}, and again compile a table (Table 3).

Table 3

Computational Table for Second Approximation ($p^2 = 700$ sec^{-2})

No. of disk	J	$Jp^2 \times 10^{-6}$	a	$Jp^2a \times 10^{-6}$	$\Sigma Jp^2a \times 10^{-6}$	$k \times 10^{-6}$	$\frac{\Sigma Jp^2a}{k}$
1	2,830	1.98	1.000	1.98	1.98	740	0.003
2	2,830	1.98	0.997	1.97	3.45	740	0.005
3	2,830	1.98	0.992	1.96	5.91	740	0.008
4	2,830	1.98	0.984	1.95	7.86	740	0.011
5	2,830	1.98	0.973	1.93	9.79	740	0.013
6	2,830	1.98	0.960	1.90	11.69	740	0.016
7	85,000	59.5	0.944	56.2	67.9	14.8	4.58
8	27,000	18.8	−3.64	−68.9	$\boxed{-1.0}$		

The residual (-1.0×10^6) has a different sign than in the first calculation; it may be stated that the true value of the square of the first frequency lies within the range from 690 to 700 sec^{-2}. Using linear interpolation, we find a sufficiently accurate value of the square of the first frequency as $p_1^2 = 694$ sec^{-2}, i.e., $p_1 = 26.4$ sec^{-1}.

There is no point in making further refinement. In Fig. 51, a dashed line indicates the refined natural mode of vibration; it differs only slightly from the mode obtained in Example 8 (solid line).

Refinement of second frequency. We fill in a computational table (Table 4) assuming $p^2 = 15.000$ sec^{-2}.

Table 4

Computational Table for First Approximation ($p^2 = 15,000$ sec^{-2})

No. of disk	J	$Jp^2 \times 10^{-6}$	a	$Jp^2a \times 10^{-6}$	$\Sigma Jp^2a \times 10^{-6}$	$k \times 10^{-6}$	$\frac{\Sigma Jp^2a}{k}$
1	2,830	42.4	1.000	42.4	42.4	740	0.057
2	2,830	42.4	0.943	40.0	84.4	740	0.111
3	2,830	42.4	0.832	35.2	117.6	740	0.159
4	2,830	42.4	0.673	28.5	146.1	740	0.198
5	2,830	42.4	0.475	20.2	166.3	740	0.225
6	2,830	42.4	0.250	10.6	176.9	740	0.239
7	85,000	1,275	0.011	14.0	190.9	14.8	12.90
8	27,000	405	−12.39	−5,220	$\boxed{-5,029}$		

The residual torque ($-5,029 \times 10^6$) is so large that the second approximation requires a sharp change in the computing value of p^2. As a second approximation we assume $p^2 = 20,000$ sec^{-2} and again fill in a computational table (Table 5).

Table 5

Computational Table for Second Approximation ($p^2 = 20,000$ sec^{-2})

No. of disk	J	$Jp^2 \times 10^{-6}$	a	$Jp^2a \times 10^{-6}$	$\Sigma Jp^2a \times 10^{-6}$	$k \times 10^{-6}$	$\frac{\Sigma Jp^2a}{k}$
1	2,830	56.6	1.000	56.6	56.6	740	0.077
2	2,830	56.6	0.923	52.2	108.8	740	0.147
3	2,830	56.6	0.776	43.9	152.7	740	0.206
4	2,830	56.6	0.570	32.3	185.0	740	0.250
5	2,830	56.6	0.320	18.1	203.1	740	0.274
6	2,830	56.6	0.056	3.2	206.3	740	0.279
7	85,000	1,700	−0.223	−379	−172.7	14.8	−11.67
8	27,000	540	11.45	6,180	$\boxed{6,007}$		

The residual torque ($6,007 \times 10^6$) is also very large but it differs in sign from the first approximation. By using linear interpolation, the square of the second frequency is found to be $p_2^2 = 17,280$ sec^{-2}

Next, it is necessary to repeat calculations until the difference between the frequencies in two successive calculations becomes sufficiently small. This will require at least two more tables to be filled in (the true value will be found as $p_2^2 = 17{,}780$ sec^{-2}, i.e., $p_2 = 133$ sec^{-1}).

Note that calculations of higher natural frequencies are always rather laborious.

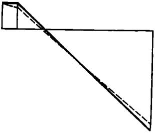


Fig. 51

Flexural Vibrations of Beams

General considerations. Flexural vibrations of multimass beam systems are widely met in engineering structures and also in turbines which incorporate straight shafts carrying a number of disks.

Consider free vibrations of an arbitrarily fixed elastic beam carrying an arbitrary finite number of point masses m_1, m_2, . . ., m_n (see, for example, Fig. 52*a*). As stated above, the method to be preferred in problems of this type is the inverse method based on the introduction of inertia forces applied to the massless elastic "framework" of the system. It is then convenient to use the concept of a *unit displacement* δ_{ik} defined as the displacement in the i direction due to a unit force acting in the k direction (Fig. 52*b*). If the system is acted on by a force $P_k \neq 1$ in the k direction and we wish to determine the total displacement produced in the i direction, we can write $y_i = P_k\delta_{ik}$ because of the proportionality between the force and displacement.

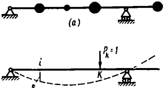


Fig. 52

For a system subjected simultaneously to forces P_1, P_2, . . ., P_n the total displacement is found by summation

$$y_i = \sum_{k=1}^{n} P_k\delta_{ik}. \tag{7.27}$$

This formula forms the basis for the derivation of equations of vibration by the inverse method [*]. The methods of determining

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[*] Unit displacements are also called *influence coefficients*. P_k may imply couples; then δ_{ik} is the displacement produced by a unit couple $P_k = 1$ in the i direction.

δ_{ik} will not be discussed in detail as they are given in texts on strength of materials; we present only some necessary fundamental data.

To determine the displacement δ_{ik} for bars subjected to bending, use is made of *Mohr's formula*

$$\delta_{ik} = \Sigma \int_0^l \frac{\overline{M}_i \overline{M}_k \, dx}{EI},$$

where EI = flexural rigidity which is a function of abscissa x in general,

$\overline{M}_i(x)$, $\overline{M}_k(x)$ = bending moments produced, respectively, by unit forces $P_i = 1$ and $P_k = 1$ (Fig. 53).

In many cases it is convenient to use *Vereshchagin's formula*, which is a semigraphical interpretation of Mohr's formula

$$\delta_{ik} = \Sigma \frac{\Omega_i \overline{M}_k^*}{EI},$$

where Ω_i = area of a portion of M_i diagram,

$\overline{M}_k^*$ = ordinate in $\overline{M}_k$ diagram located directly below centroid of Ω_i.

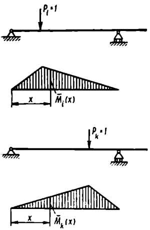

Fig. 53

When using Vereshchagin's formula, the length of the bar should be divided into portions such that the $\overline{M}_k$ diagram is bounded by a straight line within each of them.

It is well to recall also the validity of the equality $\delta_{ik} = \delta_{ki}$ which is merely Maxwell's theorem in symbols.

Basic equations of problem and frequency equation. In the case of free vibration of a beam carrying masses $m_1, m_2, \ldots, m_n$ the inertia forces $-m_1\ddot{y}_1, -m_2\ddot{y}_2, \ldots, -m_n\ddot{y}_n$ of these masses are the only load on the elastic "framework" of the system during vibration. Therefore, according to formula (7.27) we can write for any displacement

$$y_i = -\sum_{k=1}^{n} m_k \ddot{y}_k \delta_{ik} \qquad (i = 1, 2, \ldots, n)$$

or in expanded form

$$\left.\begin{aligned}
y_1 &= -m_1\ddot{y}_1\delta_{11} - m_2\ddot{y}_2\delta_{12} - \ldots - m_n\ddot{y}_n\delta_{1n},\\
y_2 &= -m_1\ddot{y}_1\delta_{21} - m_2\ddot{y}_2\delta_{22} - \ldots - m_n\ddot{y}_n\delta_{2n},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
y_n &= -m_1\ddot{y}_1\delta_{n1} - m_2\ddot{y}_2\delta_{n2} - \ldots - m_n\ddot{y}_n\delta_{nn}.
\end{aligned}\right\} \qquad (7.28)$$

The coefficients m_k appearing here have a generalized meaning since, in problems where one of the displacements, y_k, is the angle of rotation, by m_k is meant not the mass but the moment of inertia of the kth mass about a gravity axis perpendicular to the plane of vibration.

In the simple case of systems of one degree of freedom the basic set of equations (7.28) reduces to a single equation with one unknown function

$$y_1 = -m_1\ddot{y}_1\delta_{11};$$

it is equivalent to the standard equation $m\ddot{y}_1 + ky_1 = 0$ as $k = 1/\delta_{11}$.

A particular solution of (7.28) in the case of systems of several degrees of freedom may be assumed in the form

$$y_i = a_i \sin(pt + \alpha). \qquad (7.29)$$

Substituting expression (7.29) in Eqs. (7.28), we obtain a system of homogeneous algebraic equations in the unknown amplitudes

$$\left.\begin{aligned}
a_1(m_1\delta_{11}p^2 - 1) + a_2m_2\delta_{12}p^2 + \ldots + a_nm_n\delta_{1n}p^2 &= 0,\\
a_1m_1\delta_{21}p^2 + a_2(m_2\delta_{22}p^2 - 1) + \ldots + a_nm_n\delta_{2n}p^2 &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
a_1m_1\delta_{n1}p^2 + a_2m_2\delta_{n2}p^2 + \ldots + a_n(m_n\delta_{nn}p^2 - 1) &= 0.
\end{aligned}\right\} \qquad (7.30)$$

This system is satisfied by the trivial solution

$$a_1 = a_2 = \ldots = a_n = 0$$

which corresponds to the *absence of vibration* but it is not unique.

The amplitudes a_i are not all zero simultaneously if the determinant of the coefficients of the system of equations (7.30) is equal to zero

$$\begin{vmatrix}
m_1\delta_{11}p^2 - 1 & m_2\delta_{12}p^2 \ldots\ldots\ldots m_n\delta_{1n}p^2\\
m_1\delta_{21}p^2 & m_2\delta_{22}p^2 - 1 \ldots\ldots m_n\delta_{2n}p^2\\
\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\\
m_1\delta_{n1}p^2 & m_2\delta_{n2}p^2 \ldots\ldots\ldots m_n\delta_{nn}p^2 - 1
\end{vmatrix} = 0. \qquad (7.31)$$

Expanding determinant (7.31), we obtain the frequency equation of degree n for the square of the frequency p^2; we write this equation as[*]

$$1-b_1p^2+b_2p^4-b_3p^6+\ldots+(-1)^n b_np^{2n}=0. \qquad (7.32)$$

The number of roots of Eq. (7.32) is equal to n; denote these roots by $p_1^2, p_2^2, \ldots, p_n^2$ in increasing order of magnitude. All the roots are real and positive, therefore there are just as many values for the frequency p (negative solutions have no physical meaning)

$$p_1=+\sqrt{p_1^2},\quad p_2=+\sqrt{p_2^2},\ \ldots,\ p_n=+\sqrt{p_n^2}.$$

The general solution of the system of equations (7.28) is made up of the sum of particular solutions of the type of (7.29)

$$y_i=\sum_{k=1}^{n} a_{ik}\sin(p_kt+\alpha_k) \qquad (i=1,\ 2,\ \ldots,\ n) \qquad (7.33)$$

or in expanded form

$$\begin{aligned}
y_1&=a_{11}\sin(p_1t+\alpha_1)+a_{12}\sin(p_2t+\alpha_2)+\ldots+a_{1n}\sin(p_nt+\alpha_n),\\
y_2&=a_{21}\sin(p_1t+\alpha_1)+a_{22}\sin(p_2t+\alpha_2)+\ldots+a_{2n}\sin(p_nt+\alpha_n),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
y_n&=a_{n1}\sin(p_1t+\alpha_1)+a_{n2}\sin(p_2t+\alpha_2)+\ldots+a_{nn}\sin(p_nt+\alpha_n).
\end{aligned}$$

Thus, multitone vibrations occur in each of the directions $i = 1, 2, \ldots, n$ and the number of components is equal to the number of natural frequencies, i.e., to the number of degrees of freedom of the system.

We focus our attention on the calculation of the natural frequencies leaving out for a while the determination of the amplitudes a_i.

In the particular case of a system of two degrees of freedom the set of equations (7.30) becomes

$$\left.\begin{aligned}
a_1(m_1\delta_{11}p^2-1)+a_2m_2\delta_{12}p^2&=0,\\
a_1m_1\delta_{21}p^2+a_2(m_2\delta_{22}p^2-1)&=0
\end{aligned}\right\} \qquad (7.34)$$

and in place of determinant (7.31) we have

$$\begin{vmatrix} m_1\delta_{11}p^2-1 & m_2\delta_{12}p^2 \\ m_1\delta_{21}p^2 & m_2\delta_{22}p^2-1 \end{vmatrix}=0. \qquad (7.35)$$

The frequency equation (7.32) becomes biquadratic and has the following solutions

$$p_{1,2}^2=\frac{m_1\delta_{11}+m_2\delta_{22}\pm\sqrt{(m_1\delta_{11}+m_2\delta_{22})^2-4(\delta_{11}\delta_{22}-\delta_{12}^2)m_1m_2}}{2m_1m_2(\delta_{11}\delta_{22}-\delta_{12}^2)} \qquad (7.36)$$

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[*] With such arrangement of the signs, all the coefficients b_h are positive.

Example 11. Find the natural frequencies of vibration of the cantilever beam shown in Fig. 54*a* taking into account the rotatory inertia of the end mass. This system has two degrees of freedom and its motion is defined by two co-ordinates, y_1 (vertical displacement of the mass) and y_2 (angle of rotation of the mass).

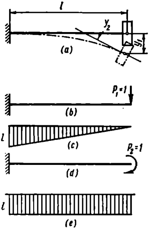


Fig. 54

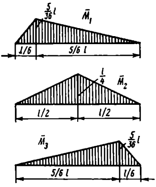


Fig. 55

Plot bending moment diagrams due to a force $P_1 = 1$ and a moment $P_2 = 1$ (Fig. 54*b* to *e*) and calculate by Vereshchagin's formula

$$\delta_{11}=\frac{l^3}{3EI}, \quad \delta_{12}=\delta_{21}=\frac{l^2}{2EI}, \quad \delta_{22}=\frac{l}{EI} .$$

Next, use formula (7.36) substituting $m_1=m$, $m_2=m\rho^2$ (ρ is the radius of gyration of the mass)

$$p_{1,2}^2=\frac{6EI}{ml^3}\frac{2EI}{m\rho^2 l}\left(1+\frac{3\rho^2}{l^2}\pm\sqrt{1+\frac{3\rho^2}{l^2}+\frac{9\rho^4}{l^4}}\right) .$$

If, as is usually the case, the ratio l/ρ is large, then

$$p_{1,2}^2=\frac{6EI}{ml^3}\left[1+\frac{l^2}{3\rho^2}\pm\left(\frac{9\rho^2}{8l^2}+\frac{1}{2}+\frac{l^2}{3\rho^2}\right)\right].$$

Taking the minus sign, we obtain the square of the lowest (first) frequency

$$p_1^2=\frac{3EI}{ml^3}\left(1-\frac{9\rho^2}{4l^2}\right) .$$

The square of the highest (second) frequency is found to be many times as large

$$p_2^2=\frac{9EI}{ml^3}\left(1+\frac{4l^2}{9\rho^2}\right) .$$

Thus, for example, if $\rho/l = 0.1$, the second frequency is 12 times the first one.

Example 12. Determine the natural frequencies of vibration of a beam carrying three concentrated masses (see Fig. 23).

To determine unit displacements we plot bending moment diagrams due to unit forces $P_1 = 1$, $P_2 = 1$ and $P_3 = 1$ (Fig. 55). By Vereshchagin's formula we find the unit displacements

$$\delta_{11} = \delta_{33} = 75k, \quad \delta_{22} = 243k, \quad \delta_{21} = \delta_{12} = \delta_{32} = \delta_{23} = 117k,$$
$$\delta_{13} = \delta_{31} = 51k.$$

In these expressions $k = l^3/(9 \times 1.296EI)$.

We now set up determinant (7.35)

$$\begin{vmatrix} 75\,mkp^2 - 1 & 117\,mkp^2 & 51\,mkp^2 \\ 117\,mkp^2 & 243\,mkp^2 - 1 & 117\,mkp^2 \\ 51\,mkp^2 & 117\,mkp^2 & 75\,mkp^2 - 1 \end{vmatrix} = 0.$$

Expanding the determinant, we obtain the frequency equation which is bicubic in this case

$$77{,}760\,(mkp^2)^3 - 12{,}096\,(mkp^2)^2 + 393\,mkp^2 - 1 = 0$$

and has three roots

$$p_1 = \frac{5.692}{l}\sqrt{\frac{EI}{ml}}, \quad p_2 = \frac{22.05}{l}\sqrt{\frac{EI}{ml}}, \quad p_3 = \frac{36.00}{l}\sqrt{\frac{EI}{ml}}$$

(the calculation of these roots is left out).

Natural modes of vibration. If any of the roots of the frequency equation is substituted in the system of equations (7.30), one of the equations will become a consequence of the others, i.e., there will be only $n - 1$ independent equations. These equations relate n amplitudes $a_{1i}, a_{2i}, \ldots, a_{ni}$ [the second subscript indicates the number of the root p_i^2 substituted in system (7.30)] and therefore allow all the amplitudes to be expressed in terms of any one of them, say, the first one. The ratios a_{2i}/a_{1i}, a_{3i}/a_{1i}, ... define the ith natural mode of vibration. Thus, to each frequency corresponds a definite mode of vibration; there are just as many such modes as there are degrees of freedom of the system.

As was done above, it may be proved that any two natural modes of vibration are mutually orthogonal

$$m_1 a_{1i} a_{1k} + m_2 a_{2i} a_{2k} + \ldots + m_n a_n\, a_{nk} = 0.$$

Example 13. Determine the natural modes of vibration of a cantilever beam (see Fig. 54). Since the system has two degrees of freedom, the number of natural modes is also two.

Substituting the relevant data

$$m_1 = m, \quad m_2 = m\rho^2, \quad \delta_{11} = \frac{l^3}{3EI}, \quad \delta_{12} = \frac{l^2}{2EI}$$

in the first equation of system (7.34)

$$a_1 (m_1 \delta_{11} p^2 - 1) + a_2 m_2 \delta_{12} p^2 = 0,$$

we obtain

$$a_1\left(\frac{ml^3}{3EI}p^2-1\right)+a_2\frac{ml^2\rho^2}{2EI}p^2=0.$$

Putting

$$p^2=p_1^2=\frac{3EI}{ml^3}\left(1-\frac{9\rho^2}{4l^2}\right),$$

we find the first natural mode

$$\frac{a_{21}}{a_{11}}=\frac{3}{2l}.$$

It is shown in the upper diagram of Fig. 56 and is primarily characterized by the linear displacement of the end mass.

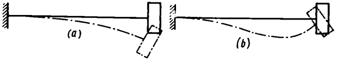


Fig. 56

To determine the second natural mode of vibration we must put $p^2=p_2^2$; hence we find

$$\frac{a_{22}}{a_{12}}\approx-\frac{2l}{3\rho^2};$$

this mode is illustrated in the lower diagram of Fig. 56 and is characterized mainly by the rotation of the end mass.

The fundamental natural mode of vibration of multimass systems can be determined by a method of successive approximations generally similar to Stodola's method presented in Sec. 4. The advantage of this method is that the calculations involve no operations with determinants.

Represent Eqs. (7.30) as

$$\left.\begin{aligned}
a_1&=p^2(m_1\delta_{11}a_1+m_2\delta_{12}a_2+\dots+m_n\delta_{1n}a_n),\\
a_2&=p^2(m_1\delta_{21}a_1+m_2\delta_{22}a_2+\dots+m_n\delta_{2n}a_n),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
a_n&=p^2(m_1\delta_{n1}a_1+m_2\delta_{n2}a_2+\dots+m_n\delta_{nn}a_n).
\end{aligned}\right\}\qquad(7.37)$$

Assigning an arbitrary set of amplitudes $(a_1)_0, (a_2)_0, \dots, (a_n)_0$ (a zero approximation to the mode of vibration), we substitute them on the right-hand sides of system (7.37) and calculate the expressions in the parentheses. Let the resulting quantities be $(b_1)_0, (b_2)_0, \dots, (b_n)_0$. Then

$$a_1=p^2(b_1)_0,\quad a_2=p^2(b_2)_0,\ \dots,\ a_n=p^2(b_n)_0.$$

These expressions may be regarded as a new approximation to the mode of vibration. Without detriment we may drop the common factor p^2 as the scale of the mode of vibration is unimportant; consequently, the first approximation to the mode of vibration is given by the amplitudes

$$(a_1)_1 = (b_1)_0, \quad (a_2)_1 = (b_2)_0, \ \ldots, \ (a_n)_1 = (b_n)_0.$$

Substituting these quantities on the right-hand sides of system (7.37), we obtain in a similar way the second approximation, etc. Carrying on this process, we may (and quite quickly) come to an approximate repetition of the results when the set of amplitudes for the $(k-1)$th approximation will differ from the set of amplitudes for the kth approximation only by the common scale factor, i.e.,

$$\frac{(a_1)_k}{(a_1)_{k-1}} \cong \frac{(a_2)_k}{(a_2)_{k-1}} \cong \ldots \cong \frac{(a_n)_k}{(a_n)_{k-1}}.$$

This completes the determination of the fundamental mode of vibration. Then we immediately find the fundamental frequency. Since

$$(a_i)_k = p^2 (a_i)_{k-1},$$

we obtain for the square of the frequency

$$p^2 = \frac{(a_i)_k}{(a_i)_{k-1}}.$$

Automobile Vibration

The automobile will be treated as a system of elastically connected rigid bodies *1* to *5* (Fig. 57*a*). Here body *1* represents schematically the automobile body, and bodies *2* to *5* represent the wheels, whose masses are assumed to be concentrated. This scheme is arrived at when considering railroad cars, locomotives and other similar transportation facilities.

The motion of this system during vibration is characterized by seven co-ordinates:

y_1 = vertical displacement of centre of gravity of automobile body,
y_2, y_3, y_4, y_5 = vertical displacements of centres of gravity of wheels,
y_6 = rotation of automobile body about lateral axis,
y_7 = rotation of automobile body about longitudinal axis.

The distribution of the masses of the automobile and of the stiffnesses of elastic connections is nearly symmetrical with respect to the middle longitudinal plane, hence a slight asymmetry is ignored in vibration calculations. The overall process of vibration can then be considered as consisting of two independent processes (Fig. 57*b* and *c*):

longitudinal vibrations which are characterized by a vertical displacement of the body (y_1), a rotation of the body about the lateral axis (y_6) and pairwise equal displacements of the two front wheels ($y_2 = y_4$) and the two rear wheels ($y_3 = y_5$);

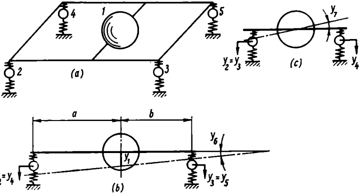


Fig. 57

transverse (lateral) vibrations which are characterized by a rotation of the body about the longitudinal axis (y_7) and pairwise equal displacements of the two left wheels ($y_2 = y_3$) and the two right wheels ($y_4 = y_5$).

Accordingly the longitudinal vibrations are defined by four differential equations and the transverse vibrations by three. Consider the longitudinal vibrations which are of prime importance.

Denote the stiffnesses of the tires by k, the stiffnesses of the front and rear springs by k_f and k_r, respectively, the masses of the body and of the wheel by m and m_w, respectively. The radius of gyration of the body about a lateral gravity axis is denoted by ρ. In this notation the deformations of the springs are

$$\Delta_f = y_1 + ay_6 - y_2 \qquad \text{(front spring)},$$

$$\Delta_r = y_1 - by_6 - y_3 \qquad \text{(rear spring)}.$$

The equations of motion will be written in Lagrangian form. The kinetic energy of the system is made up of the following components:

$$\frac{m\dot{y}_1^2}{2}+\frac{m\rho^2\dot{y}_6^2}{2}=\text{kinetic energy of body}$$

$$2\,\frac{m_w\dot{y}_2^2}{2}=\text{kinetic energy of front wheels}$$

$$2\,\frac{m_w\dot{y}_3^2}{2}=\text{kinetic energy of rear wheels}$$

The total kinetic energy is

$$T=\frac{1}{2}\left[m\left(\dot{y}_1^2+\rho^2\dot{y}_6^2\right)+2m_w\left(\dot{y}_2^2+\dot{y}_3^2\right)\right]. \tag{7.38}$$

The potential energy consists of the strain energy of the springs

$$2\,\frac{k_f\Delta_f^2}{2}+2\,\frac{k_r\Delta_r^2}{2}=k_f\left(y_1-y_2+ay_6\right)^2+k_r\left(y_1-y_3-by_6\right)^2$$

and the energy of the compressed tires

$$2\,\frac{ky_2^2}{2}+2\,\frac{ky_3^2}{2}=k\left(y_2^2+y_6^2\right).$$

The total potential energy is

$$U=k_f\left(y_1-y_2+ay_6\right)^2+k_r\left(y_1-y_3-by_6\right)^2+k\left(y_2^2+y_3^2\right). \tag{7.39}$$

In this case Lagrange's equations are of the form

$$\left.\begin{aligned}m\ddot{y}_1+2k_f\left(y_1-y_2+ay_6\right)+2k_r\left(y_1-y_3-by_6\right)&=0,\\ 2m_w\ddot{y}_2-2k_f\left(y_1-y_2+ay_6\right)+2ky_2&=0,\\ 2m_w\ddot{y}_3-2k_r\left(y_1-y_3+by_6\right)+2ky_3&=0,\\ m\rho^2\ddot{y}_6+2k_f\left(y_1-y_2+ay_6\right)a-2k_r\left(y_1-y_3-by_6\right)b&=0.\end{aligned}\right\} \tag{7.40}$$

A particular solution of this system is

$$y_i=a_i\sin\left(pt+\alpha\right)\qquad(i=1,\ 2,\ 3,\ 6). \tag{7.41}$$

The substitution of expression (7.4) in the system of equations (7.40) leads to an equation of the same structure as in the problems considered above; thus, corresponding to the number of degrees of freedom there will be four values of the natural frequency of vibration.

Assuming the tires to be undeformable, consider a *simplified scheme* of longitudinal vibrations (Fig. 58); this system has two degrees of freedom corresponding to the coordinates y_1 and y_6. We put $y_2=y_3=0$ in expressions (7.38) and (7.39). We then obtain for the kinetic and potential energy, respectively,

$$T=\frac{m}{2}(\dot{y}_1^2+\rho^2\dot{y}_6^2),$$

$$U=k_f(y_1+ay_6)^2+k_r(y_1-by_6)^2.$$

Lagrange's equations become

$$m\ddot{y}_1+2k_f(y_1+ay_6)+2k_r(y_1-by_0)=0,$$

$$m\rho^2\ddot{y}_6+2k_f(y_1+ay_6)\,a-2k_r(y_1-by_0)\,b=0.$$

Substituting the particular solution (7.41), we have the system

$$-ma_1p^2+2k_f(a_1+aa_6)+2k_r(a_1-ba_6)=0,$$

$$-m\rho^2p_2a_6+2k_f(a_1+aa_6)\,a-2k_r(a_1-ba_6)\,b=0$$

or

$$\left.\begin{aligned}a_1(-mp^2+2k_f+2k_r)+a_6(2k_fa-2k_rb)=0,\\ a_1(2k_fa-2k_rb)+a_6(-m\rho^2p^2+2k_fa^2+2k_rb^2)=0.\end{aligned}\right\}\qquad(7.42)$$

A non-trivial solution is found by equating the determinant of

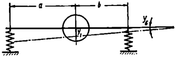


Fig. 58

the coefficients of a_1 and a_6 to zero

$$\begin{vmatrix}-mp^2+2k_f+2k_r & 2k_fa-2k_rb\\ 2k_fa-2k_rb & -m\rho^2p^2+2k_fa^2+2k_rb^2\end{vmatrix}=0.$$

Expanding the determinant, we obtain the frequency equation of the second degree in p^2

$$p^4-\frac{2}{m\rho^2}p^2[k_f(a^2+\rho^2)+k_r(b^2+\rho^2)]+\frac{4k_fk_r}{m^2\rho^2}\quad(a+b)^2=0.\quad(7.43)$$

After determining the frequencies from this equation, we can find both natural modes of vibration. To do this, it is necessary to form the amplitude ratio using either one (say, the first) of the equations of system (7.42)

$$\frac{a_6}{a_1}=\frac{mp^2-2k_f-2k_r}{2k_fa-2k_rb}\qquad(7.44)$$

and substitute both roots of the frequency equation successively; the same result can be obtained using the second equation of (7.42).

Consider the particular case of mass distribution when $\rho^2 = ab$. In this case the roots of the frequency equation (7.43) are

$$\left.\begin{aligned} p_1^2 &= \frac{2k_f(a+b)}{mb}, \\ p_2^2 &= \frac{2k_r(a+b)}{ma}. \end{aligned}\right\} \qquad (7.45)$$

To determine the natural modes of vibration, we substitute these roots in expression (7.44) successively. We then obtain for the first mode

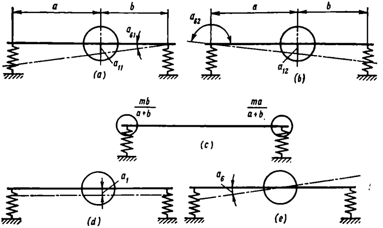


Fig. 59

$$a_{\theta 1} = \frac{a_{11}}{b}$$

and for the second mode

$$a_{\theta 2} = -\frac{a_{12}}{a}.$$

These modes are shown in Fig. 59*a* and *b*; an essential feature is that one of the automobile axles has no motion when the other is vibrating. Formulas (7.45) indicate that in this particular case the frequencies can be calculated using the diagram shown in Fig. 59*c*, i.e., having the total mass distributed according to a lever law.

In another particular case when $k_f a = k_r b$, Eqs. (7.42) become independent

$$\left.\begin{aligned} a_1(-mp^2+2k_f+2k_r)&=0,\\ a_6(-m\rho^2p^2+2k_fa^2+2k_rb^2)&=0.\end{aligned}\right\} \qquad (7.46)$$

This points to the possibility for purely vertical vibrations involving no rotation ("*bouncing*"—see Fig. 59*d*) and also for purely angular vibrations with the centre of gravity fixed ("*galloping*"—see Fig. 59*e*). Indeed, system (7.46) is satisfied by the solution $a_1 \neq 0$, $a_6 = 0$ if

$$-mp^2+2k_f+2k_r=0 \qquad (7.47)$$

and by the solution $a_1=0$, $a_6 \neq 0$ if

$$-m\rho^2p^2+2k_fa^2+2k_rb^2=0. \qquad (7.48)$$

From equality (7.47) we find the first natural frequency

$$p_1=\sqrt{\frac{2(k_f+k_r)}{m}}$$

and from equality (7.48), the second natural frequency

$$p_2=\sqrt{\frac{2(k_fa^2+k_rb^2)}{m\rho^2}}.$$

Example 14. Determine the natural frequencies and the natural modes of vibration of an automobile for which the following data are given: $m = 1.6$ kgf-sec^2-cm^{-1}, $\rho = 122.5$ cm, $2k_f = 48.4$ kgf-cm^{-1}, $2k_r = 37$ kgf-cm^{-1}, $a = 131$ cm, $b = 139$ cm.

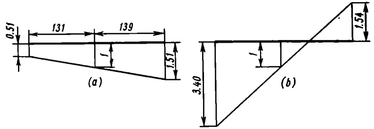


Fig. 60

Equation (7.43) becomes

$$p^4-117.8p^2+3{,}400=0,$$

whence

$$p^2=(58.9 \pm 8.3)\ \text{sec}^{-2}.$$

The frequencies are: $p_1 = 7.11\ \text{sec}^{-1}$ and $p_2 = 8.20\ \text{sec}^{-1}$.
To determine the natural modes of vibration we use formula (7.44)

$$\frac{a_{\varphi 1}}{a_{11}} = \frac{1.6 \times 50.6 - 48.4 - 37}{48.4 \times 131 - 37 \times 139} = -0.0037\ \text{cm}^{-1},$$

$$\frac{a_{\varphi 2}}{a_{12}} = \frac{1.6 \times 67.2 - 48.4 - 37}{48.4 \times 131 - 37 \times 139} = 0.0183\ \text{cm}^{-1}.$$

These modes are shown in Fig. 60*a* and *b*; the first mode constitutes primarily "bouncing" of the body, and the second, "galloping".

Check the orthogonality of these modes. In our case the orthogonality condition is

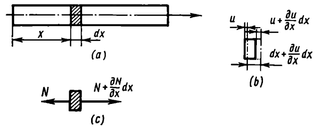
$$ma_{11}a_{12} + m\rho^2 a_{\varphi 1}a_{\varphi 2} = ma_{11}a_{12}\left(1 + \rho^2 \frac{a_{\varphi 1}}{a_{11}} \frac{a_{\varphi 2}}{a_{12}}\right) = 0.$$

Substituting the values, we obtain

$$ma_{11}a_{12}(1 - 15{,}000 \times 0.0037 \times 0.0183) \simeq 0.$$

8. Vibrations of Bars of Uniform Section (Exact Solution)

An essential feature of the process of free vibration of systems with continuously distributed mass is that there is an infinite number of natural frequencies and modes of vibration.



x
dx
(a)
u
$u+\frac{\partial u}{\partial x}dx$
$dx+\frac{\partial u}{\partial x}dx$
(b)
N
$N+\frac{\partial N}{\partial x}dx$
(c)

Fig. 61

Hence the peculiarities of a mathematical nature: instead of ordinary differential equations governing the processes in systems of several degrees of freedom we have to deal here with partial differential equations. Aside from the *initial conditions* defining the initial displacements and velocities, it is necessary to take into account the *boundary conditions* characterizing the mode of fixing the ends.

Longitudinal Vibrations of Bars

Basic equation and its solution. In analysing longitudinal vibrations of a straight bar (Fig. 61*a*) we shall assume that the cross sections remain plane and the particles of the bar do not execute any transverse motions but are displaced only in the longitudinal direction.

Let u be the longitudinal displacement of the current section of the bar during vibration; this displacement depends on the location of the section (co-ordinate x) and the time t. Thus, $u = u(x, t)$ is a function of two variables; its determination constitutes the main problem. The displacement of an infinitely close section is $u + \frac{\partial u}{\partial x} dx$ and hence the absolute elongation of an infinitesimal element dx is $\frac{\partial u}{\partial x} dx$ (Fig. 61*b*) and the unit elongation is $\varepsilon = \frac{\partial u}{\partial x}$.

Accordingly, the longitudinal force at a section of coordinate x may be written as

$$N = EA\varepsilon = EA\frac{\partial u}{\partial x}, \tag{8.1}$$

where EA is the stiffness of the bar in tension or compression. The force N is also a function of the same arguments, the co-ordinate x and the time t.

Consider an element of the bar included between two infinitely close sections (Fig. 61*c*). The left-hand face of the element is acted on by a force N and the right-hand face by a force $N + \frac{\partial N}{\partial x} dx$. If the density of the material of the bar is denoted by ρ, the mass of the element being considered is $\rho A\, dx$. Thus, the equation of motion in the x direction becomes

$$-N + \left(N + \frac{\partial N}{\partial x} dx\right) = \rho A\, dx \frac{\partial^2 u}{\partial t^2}$$

or

$$\frac{\partial N}{\partial x} = \rho A \frac{\partial^2 u}{\partial t^2}. \tag{8.2}$$

Taking into account expression (8.1) and assuming that $A = \text{constant}$, we obtain

$$k^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \tag{8.3}$$

where

$$k^2 = \frac{E}{\rho}. \tag{8.4}$$

Following Fourier's method, we seek a particular solution of the differential equation (8.3) in the form

$$u + X(x)\,T(t), \tag{8.5}$$

i.e., we assume that the displacement u can be represented as the product of two functions one of which depends only on the argument x while the other depends only on the argument t. Instead of determining the function of two variables $u(x, t)$ it is then necessary to determine two functions, $X(x)$ and $T(t)$, each of which depends only on one variable.

Substituting expression (8.5) in Eq. (8.3), we obtain $k^2X''T = X\ddot{T}$, where primes indicate differentiation with respect to x. Rewrite this equation as

$$\frac{k^2X''}{X} = \frac{\ddot{T}}{T}.$$

Here the left-hand side depends only on x while the right-hand side depends only on t; since this equality must be fulfilled identically (for all values of x and t), each side must be equal to a constant which will be denoted by $-p^2$

$$\frac{k^2X''}{X} = -p^2, \quad \frac{\ddot{T}}{T} = -p^2. \tag{8.5a}$$

Hence we have two equations

$$\ddot{T} + p^2T = 0, \quad X'' + \frac{p^2}{k^2}X = 0. \tag{8.6}$$

The first equation has the solution

$$T = a\sin(pt + \alpha) \tag{8.7}$$

indicating the vibratory nature of the process. From expression (8.7) it is seen that the quantity p, which is as yet unknown, has the meaning of the frequency of free vibration.

The second equation of (8.6) has the solution

$$X = C\sin\frac{p}{k}x + D\cos\frac{p}{k}x \tag{8.8}$$

and determines the mode of vibration. As will be shown later, the frequency equation defining the quantity p is derived by using the boundary conditions; this equation is always transcendental and has an infinite number of roots. Thus, the number of natural frequencies is infinite, to each value of the frequency, p_n, corresponding its own function $T_n(t)$ defined by relation (8.7) and its own function $X_n(x)$ defined by relation (8.8).

Solution (8.5) is only a particular solution and does not completely define the motion. The complete solution is obtained by superposing all the particular solutions

$$u = \sum_{n=1}^{\infty} X_n(x) T_n(t).$$

The functions $X_n(x)$ are called the *eigenfunctions* of the problem and define the natural modes of vibration. They are independent of the initial conditions and satisfy the orthogonality condition which, when A = constant, is

$$\int_0^l X_m(x) X_n(x)\,dx = 0 \text{ if } m \neq n.$$

Boundary conditions. Consider some typical cases.

Fixed end of a bar (Fig. 62*a*). The displacement u at the end section must be zero; it follows that at this section

$$X = 0. \qquad (8.9)$$

Free end of a bar (Fig. 62*b*). The longitudinal force at the end section

$$N = EAX'T \qquad (8.10)$$

must be identically zero, which is possible if at the end section

$$X' = 0.$$

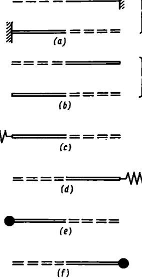


Fig. 62

Elastically restrained end of a bar (Fig. 62*c*). Due to the displacement u of the end section there is an elastic reaction of the support—$k_0 u = -k_0 XT$ (k_0 is the stiffness of the support). Taking into account expression (8.10) for the longitudinal force, we obtain the boundary condition as

$$k_0 X = EAX'$$

if the support is at the left end of the bar (Fig. 62*c*), and

$$-k_0 X = EAX'$$

if the support is at the right end of the bar (Fig. 62*d*).

Concentrated mass m_0 at the end of a bar. The inertia force developed by the mass is

$$-m_0\frac{\partial^2 u}{\partial t^2} = -m_0 X\ddot{T}.$$

Since, according to Eq. (8.6), $\ddot{T} = -p^2T$, the inertia force can be written as m_0p^2XT. Taking into account expression (8.10), we obtain the boundary condition as

$$-m_0p^2X = EAX'$$

if the mass is at the left end (Fig. 62*e*), and

$$m_0p^2X = EAX' \qquad (8.11)$$

if the mass is attached to the right end (Fig. 62*f*).

Frequency equation. Consider some typical special cases.

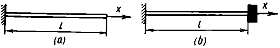


Fig. 63

Determination of natural frequencies of the cantilever beam shown in Fig. 63a. According to expressions (8.9) and (8.10)

$$X = 0 \quad \text{at} \quad x = 0, \qquad X' = 0 \quad \text{at} \quad x = l.$$

Substituting these conditions in solution (8.8) successively, we obtain

$$D = 0, \quad C\frac{p}{k}\cos\frac{pl}{k} = 0.$$

The condition $C \neq 0$ leads to the frequency equation

$$\cos\frac{pl}{k} = 0.$$

The roots of this equation

$$\frac{pl}{k} = (2n-1)\frac{\pi}{2} \qquad (n = 1,\ 2,\ \ldots)$$

determine the natural frequencies

$$p_n = \frac{(2n-1)\pi k}{2l} \qquad (n = 1,\ 2,\ \ldots). \qquad (8.12)$$

The number of frequencies is infinite; the first (lowest) frequency, for $n = 1$, is

$$p_1 = \frac{\pi k}{2l} = \frac{\pi}{2l}\sqrt{\frac{E}{\rho}}.$$

The second frequency, for $n = 2$, is

$$p_2 = \frac{3\pi}{2l}\sqrt{\frac{E}{\rho}}$$

and so on.

Determination of natural frequencies of the beam shown in Fig. 63b. According to expressions (8.9) and (8.11) we have

$$X = 0 \text{ at } x = 0, \; m_0 p^2 X = EAX' \text{ at } x = l.$$

Substituting these conditions in solution (8.8), we obtain

$$D = 0, \; m_0 p^2 \sin\frac{pl}{k} = \frac{EAp}{k}\cos\frac{pl}{k}.$$

Consequently, the frequency equation, with expression (8.4) taken into account, is

$$\frac{pl}{k}\tan\frac{pl}{k} = \frac{\rho Al}{m_0};$$

the right-hand side of the equation represents the ratio of the mass of the beam to the concentrated end mass.

The solution of the above transcendental equation calls for the use of one of the approximate methods. The values of the most important (lowest) root of the equation $p_1 l/k$ as a function of the ratio $\alpha = \rho Al/m_0$ are the following:

α	... 0.10	0.30	0.50	0.70	0.90	1.00	2.00	4.00	10.00
$p_1 l/k$	... 0.32	0.52	0.65	0.75	0.82	0.86	1.08	1.27	1.42

When the ratio α is small, the concentrated mass is the deciding factor and good results are obtained with the approximate solution

$$p_1 = \sqrt{\frac{EA}{m_0 l\left(1 + \frac{\alpha}{3}\right)}}.$$

Bars of variable section. When $A \neq$ constant, we have from Eqs. (8.1) and (8.2)

$$k^2 \frac{\partial}{\partial x}\left(A\frac{\partial u}{\partial x}\right) = A\frac{\partial^2 u}{\partial t^2}.$$

This differential equation does not yield to solution in closed form. Hence, in such cases recourse is made to approximate methods of determining the natural frequencies (see Sec. 9).

Torsional Vibrations of Shafts

Torsional vibrations of a shaft with continuously distributed mass (Fig. 64*a*) are governed by equations which are exactly the same in structure as the equations presented above.

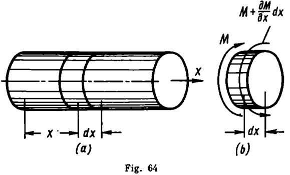


Fig. 64

Basic equation and its solution. The twisting moment M at a section with abscissa x is related to the angle of rotation φ by a differential relation similar to expression (8.1)

$$M=GI_p\frac{d\varphi}{dx}, \tag{8.13}$$

where I_p is the polar moment of inertia of the cross section.

The twisting moment at an adjacent section is $M+\frac{\partial M}{\partial x}dx$ (Fig. 64*b*).

Denoting by ρI_p (where ρ is the density of the shaft material) the intensity of the mass moment of inertia of the shaft about its axis (i.e., the moment of inertia per unit length), the equation of motion for the elementary portion of the shaft can be written as

$$-M+\left(M+\frac{\partial M}{\partial x}\right)dx=\rho I_p\frac{\partial^2\varphi}{\partial t^2}dx$$

or, similarly to Eq. (8 2),

$$\frac{\partial M}{\partial x}=\rho I_p\frac{\partial^2\varphi}{\partial t^2}.$$

Substituting expression (8.13), we obtain for $I_p=$ constant, similarly to Eq. (8.3),

$$k_1^2\frac{\partial^2\varphi}{\partial x^2}=\frac{\partial^2\varphi}{\partial t^2}, \tag{8.14}$$

where

$$k_1^2 = \frac{G}{\rho} .$$

The general solution of Eq. (8.14), as for Eq. (8.3), is of the form

$$\varphi = \sum_{n=1}^{\infty} X_n(x)\, T_n(t)$$

with

$$X_n(x) = C_n \sin \frac{p_n x}{k_1} + D_n \cos \frac{p_n x}{k_1} \qquad (8.15)$$

and

$$T_n(t) = a \sin(p_n t + \alpha_n).$$

The natural frequencies and eigenfunctions are determined by the prescribed boundary conditions.

Boundary conditions. For the fundamental modes of fixing the ends, similarly to the case of longitudinal vibrations, we obtain the following:

fixed end ($\varphi = 0$) $X = 0$; free end ($M = 0$) $X' = 0$; elastically restrained left end (k_0 is the spring stiffness) $k_0 X = GI_p X'$; elastically restrained right end $-k_0 X = GI_p X'$; disk at left end (J_0 is the moment of inertia of the disk about the axis of the shaft) $-J_0 p^2 X = GI_p X'$; disk at right end $J_0 p^2 X = GI_p X$.

Frequency equation. If the shaft is fixed at the left end ($x = 0$) and the right end ($x = l$) is free, then $X = 0$ at $x = 0$ and $X' = 0$ at $x = l$; the result is similar to expression (8.12)

$$p_n = \frac{(2n-1)\pi k_1}{2l} \qquad (n = 1,\ 2,\ \ldots).$$

If the left end is fixed and the right end carries a disk, we obtain the previous transcendental equation

$$\frac{pl}{k_1} \tan \frac{pl}{k_1} = \frac{\rho I_p l}{J_0} .$$

If both ends of the shaft are fixed, the boundary conditions are: $X = 0$ when $x = 0$ and $x = l$.

In this case we obtain from expression (8.15)

$$C \sin \frac{pl}{k_1} = 0, \quad D = 0,$$

i.e.,

$$\frac{pl}{k_1} = n\pi \qquad (n = 1,\ 2,\ \ldots)$$

and from this we find the frequencies

$$p_n = \frac{n\pi k_1}{l} .$$

If the left end of the shaft is free and the right end carries a disk, then $X' = 0$ at $x = 0$; $J_0 p^2 X = GI_p X'$ at $x = l$. Using expression (8.15) we find

$$C = 0, \quad J_0 p^2 \cos \frac{pl}{k_1} = -GI_p \frac{p}{k_1} \sin \frac{pl}{k_1}$$

or

$$\left[\tan \frac{pl}{k_1}\right] \Big/ \frac{pl}{k_1} = -\frac{J_0}{\rho I_p l} \, .$$

Flexural Vibrations of Beams

Basic equation. From a course in strength of materials we know the relation

$$EI \frac{\partial^2 y}{\partial x^2} = M, \tag{8.16}$$

where

$EI =$ flexural rigidity

$y = y(x, t) =$ deflection

$M = M(x, t) =$ bending moment

Furthermore,

$$\frac{\partial^2 M}{\partial x^2} = q, \tag{8.17}$$

where $q =$ intensity of distributed load.

Combining expressions (8.16) and (8.17), we obtain

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) = q. \tag{8.18}$$

In the problem of free vibrations, the distributed inertia forces constitute the load on the elastic "framework"

$$q = -m \frac{\partial^2 y}{\partial t^2} \, ,$$

where m is the intensity of the mass of the beam (mass per unit length); Eq. (8.18) becomes

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) + m \frac{\partial^2 y}{\partial t^2} = 0.$$

In the particular case of uniform cross section when $EI =$ constant, $m =$ constant, we obtain

$$\frac{\partial^2 y}{\partial t^2} + \frac{EI}{m} \frac{\partial^4 y}{\partial x^4} = 0.^* \tag{8.19}$$

---

* The weight of the beam itself also produces displacements which are time independent and are neglected here. This means that the deflections are measured from the position of static equilibrium during vibration.

Solution of equation. To solve Eq. (8.19) we assume, as before,

$$y = X(x)\,T(t). \tag{8.20}$$

Then

$$\frac{\ddot{T}}{T} = -\frac{EI}{m}\frac{X^{\mathrm{IV}}}{X}.$$

Since this equality must be fulfilled identically, each of the sides of the equality must be equal to a constant. Denoting this constant by $-p^2$, we obtain two equations

$$\ddot{T} + p^2 T = 0, \tag{8.21}$$

$$X^{\mathrm{IV}} - \frac{mp^2}{EI} X = 0. \tag{8.22}$$

The first equation indicates that the motion is a vibratory one with frequency p. The second equation defines the mode of vibration. The solution of Eq. (8.22) contains four constants, according to its order,

$$X = C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx,$$

where

$$k = \sqrt[4]{\frac{mp^2}{EI}}. \tag{8.23}$$

The following alternate way of writing the general solution may be found convenient

$$X = C_1 S + C_2 T + C_3 U + C_4 V, \tag{8.24}$$

where

$$\left.\begin{aligned} S &= \frac{1}{2}(\cosh kx + \cos kx), \\ T &= \frac{1}{2}(\sinh kx + \sin kx), \\ U &= \frac{1}{2}(\cosh kx - \cos kx), \\ V &= \frac{1}{2}(\sinh kx - \sin kx) \end{aligned}\right\} \tag{8.25}$$

are *Krylov's functions*.

It should be noted that $S = 1$, $T = U = V = 0$ at $x = 0$. The functions S, T, U, V are related as follows:

$$\left.\begin{aligned} S &= \frac{1}{k} T', \\ T &= \frac{1}{k} U', \\ U &= \frac{1}{k} V', \\ V &= \frac{1}{k} S'. \end{aligned}\right\} \tag{8.26}$$

Hence the derivatives of expression (8.24) are written as

$$\left.\begin{aligned} X' &= k\,(C_1V + C_2S + C_3T + C_4U),\\ X'' &= k^2\,(C_1U + C_2V + C_3S + C_4T),\\ X''' &= k^3\,(C_1T + C_2U + C_3V + C_4S). \end{aligned}\right\} \qquad (8.27)$$

In problems of the class here considered the number of natural frequencies p_n is infinite; to each of them corresponds its own function of time T_n and its own eigenfunction X_n. The general solution is obtained by superposing the particular solutions of the form of (8.20)

$$y = \sum_{n=1}^{\infty} X_n(x)\,T_n(t). \qquad (8.28)$$

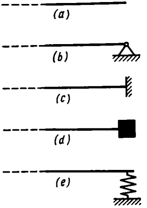


Fig. 65

The determination of natural frequencies and modes calls for the consideration of boundary conditions.

Boundary conditions. Two boundary conditions may be specified for each end of a beam.

Free end of a beam (Fig. 65*a*). The shearing force $Q = EIX'''T$ and the bending moment $M = EIX''T$ are zero. The boundary conditions are therefore

$$X'' = 0, \quad X''' = 0. \qquad (8.28a)$$

Simply supported end of a beam (Fig. 65*b*). The deflection $y = XT$ and the bending moment $M = EIX''T$ are zero. Consequently, the boundary conditions are

$$X = 0, \quad X'' = 0. \qquad (8.29)$$

Fixed end of a beam (Fig. 65*c*). The deflection $y = XT$ and the angle of rotation $\varphi = X'T$ are zero. The boundary conditions are

$$X = 0, \quad X' = 0. \qquad (8.30)$$

Concentrated mass m_0 at the end of a beam (Fig. 65*d*). Its inertia force $-m_0\ddot{y} = -m_0X\ddot{T}$ can be written as m_0p^2XT with the aid of Eq. (8.21); it must be equal to the shearing force $Q = EIX'''T$ and hence the boundary conditions become

$$m_0p^2X = \pm EIX''', \quad X'' = 0. \qquad (8.30a)$$

In the first condition, the plus sign is chosen when the concentrated mass is attached to the left end of the beam, and the minus sign when it is attached to the right end. The second condition follows from the fact that there is no bending moment.

Elastically restrained end of a beam (Fig. 65e). Here the bending moment is zero, and the shearing force $Q = EIX'''T$ is equal to the reaction of the support $-k_0 y = -k_0 XT$ (k_0 is the spring stiffness of the support).

The boundary conditions are

$$X'' = 0, \quad EIX''' = \pm k_0 X \tag{8.31}$$

(the minus sign is chosen when the elastic support is at the left end, and the plus sign when it is at the right end).

Frequency equation and natural modes. When written in expanded form the boundary conditions lead to homogeneous equations in the constants C_1, C_2, C_3 and C_4.

These constants are different from zero only if the determinant of the coefficients of the system is equal to zero; this leads to the frequency equation. As a result of these operations, we find relations between C_1, C_2, C_3 and C_4, giving the natural modes (apart from a constant factor).

The derivation of frequency equations will be illustrated by examples.

For a *beam with simply supported ends* we have, according to expressions (8.29), the following boundary conditions: $X = 0$, $X'' = 0$ when $x = 0$ and $x = l$. Using expressions (8.24) to (8.27), we obtain from the first two conditions $C_1 = 0$, $C_3 = 0$.

The two other conditions can now be written as

$$C_2 T(kl) + C_4 V(kl) = 0,$$
$$C_2 V(kl) + C_4 T(kl) = 0.$$

In order to have C_2 and C_4 different from zero, the determinant must be zero

$$\begin{vmatrix} T(kl) & V(kl) \\ V(kl) & T(kl) \end{vmatrix} = 0.$$

Thus, the frequency equation is of the form

$$[T(kl) + V(kl)]\,[T(kl) - V(kl)] = 0.$$

Substituting the expressions for T and V, we obtain

$$\sinh kl \sin kl = 0.$$

Since $\sinh kl \neq 0$, the frequency equation becomes

$$\sin kl = 0. \tag{8.32}$$

The roots of this equation are

$$kl = n\pi \qquad (n = 1,\ 2,\ \ldots).$$

Taking into account expression (8.23), we find

$$p_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{m}}\,. \tag{8.32a}$$

We proceed to the determination of the natural modes. From the above homogeneous equations the following relation between the constants C_2 and C_4 is obtained

$$C_4 = -\frac{V(kl)}{T(kl)}\,C_2.$$

Consequently, Eq. (8.24) becomes

$$X_n = C_2\left[T(k_nx) - \frac{V(k_nl)}{T(k_nl)}\,V(k_nx)\right]$$

or

$$X_n = \frac{C_2}{2T(k_nl)}\,(\sin k_nx \sinh k_nx + \sin k_nl \sinh k_nx).$$

According to condition (8.32) we have

$$X_n = C_n \sin k_nx, \tag{8.32b}$$

where $C_n = C_2 \sinh k_nl/2T(k_nl)$ is a new constant (its value remains undetermined until the initial conditions are introduced into consideration).

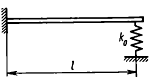


Fig. 66

Consider now a *beam with one end fixed and the other elastically restrained* (Fig. 66). According to expressions (8.30) and (8.31) the boundary conditions are: $X = 0$, $X' = 0$ when $x = 0$; $X'' = 0$, $EIX''' = k_0X$ when $x = l$. From the first two conditions it follows that $C_1 = C_2 = 0$. Using the other two conditions and expressions (8.24) and (8.27), we obtain the equations

$$C_3S(kl) + C_4T(kl) = 0,$$

$$EI\,[C_3V(kl) + C_4S(kl)] = k_0\,[C_3U(kl) + C_4V(kl)]$$

or after substitution of formulas (8.25)

$$\sinh kl \cos kl - \sinh kl \cosh kl = \frac{EI}{k_0}\,.$$

Solutions for some other cases may be found in handbooks.

Determination of motion from initial conditions. If it is desired to determine the motion resulting from an initial disturbance, it is necessary to indicate both the initial displacements and the initial velocities for all points of the beam

$$\left.\begin{aligned} y(x,\ 0) &= y_0(x), \\ v(x,\ 0) = \dot{y}(x,\ 0) &= v_0(x) \end{aligned}\right\} \tag{8.33}$$

and to use the orthogonality property of the natural modes

$$\int_0^l X_n(x)\, X_m(x)\, dx = 0 \qquad (m \neq n).$$

The general solution (8.28) is written as

$$y = \sum_{n=1}^{\infty} X_n(x)\,(A_n \sin p_n t + B_n \cos p_n t). \tag{8.34}$$

The velocity is given by the expression

$$\dot{y} = \sum_{n=1}^{\infty} p_n X_n(x)\,(A_n \cos p_n t - B_n \sin p_n t). \tag{8.35}$$

Substituting $t=0$ on the right-hand sides of Eqs. (8.34) and (8.35) and the initial displacements and velocities, which are assumed to be known, on the left-hand sides, we obtain

$$y_0 = \sum_{n=1}^{\infty} B_n X_n(x), \qquad v_0 = \sum_{n=1}^{\infty} p_n A_n X_n(x).$$

Multiplying these expressions by X_m and integrating throughout the length, we obtain

$$\left.\begin{aligned} \int_0^l y_0(x)\, X_m(x)\, dx &= B_m \int_0^l [X_m(x)]^2\, dx, \\ \int_0^l v_0(x)\, X_m(x)\, dx &= A_m p_m \int_0^l [X_m(x)]^2\, dx. \end{aligned}\right\} \tag{8.36}$$

The infinite sums on the right-hand sides have disappeared due to the orthogonality property. From expressions (8.36) the following formulas are obtained for the constants A_m and B_m

$$\left.\begin{aligned} A_m &= \frac{\int_0^l v_0(x)\, X_m(x)\, dx}{p_m \int_0^l [X_m(x)]^2\, dx}, \\ B_m &= \frac{\int_0^l y_0(x)\, X_m(x)\, dx}{\int_0^l [X_m(x)]^2\, dx}. \end{aligned}\right\} \tag{8.37}$$

It now remains to substitute these results in solution (8.34).

We note again that the choice of the scale of natural modes is unimportant. If, for example, we take C_n in the expression for the natural mode (8.32*b*) α times larger, formulas (8.37) will give results α times smaller; after substitution in solution (8.34) the factor α cancels out. Nevertheless, it is customary to use *normed* eigenfunctions choosing their scale so as to make the denominators of expressions (8.37) equal to unity; this simplifies the expressions for A_m and B_m.

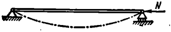


Fig. 67

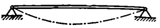


Fig. 68

Effect of a constant longitudinal force. Consider the case when a vibrating beam is subjected to a longitudinal force N whose magnitude remains constant during vibration (Fig. 67). As is known from strength of materials, for a beam with a longitudinal force the static-bending equation (8.18) is complicated and takes the form*

$$(EIy'')'' + Ny'' = q.$$

Putting $q = -m\ddot{y}$ and assuming the flexural rigidity to be constant, we obtain the equation of free vibration

$$\frac{\partial^2 y}{\partial t^2} + \frac{N}{m}\frac{\partial^2 y}{\partial x^2} + \frac{EI}{m}\frac{\partial^4 y}{\partial x^4} = 0. \tag{8.38}$$

Assume, as before, a particular solution in the form $y = X(x)T(t)$. Equation (8.38) then falls into the following equations

$$\frac{\ddot{T}}{T} = -p^2,$$

$$-\frac{N}{m}\frac{X''}{X} - \frac{EI}{m}\frac{X^{IV}}{X} = -p^2.$$

The first equation expresses the vibratory nature of the solution, while the second equation defines the mode of vibration and can also be used to find the frequencies. We rewrite it as

$$X^{IV} + \alpha^2 X'' - k^4 X = 0, \tag{8.39}$$

where k is given, as above, by formula (8.23) and

$$\alpha = \sqrt{\frac{N}{EI}}. \tag{8.40}$$

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* Here a compressive force is considered positive.

The solution of Eq. (8.39) is of the form

$$X = C_1 \sinh s_1 x + C_2 \cosh s_1 x + C_3 \sin s_2 x + C_4 \cos s_2 x,$$

where

$$s_1 = \sqrt{-\frac{\alpha^2}{2} + \sqrt{\frac{\alpha^4}{4} + k^4}},$$

$$s_2 = \sqrt{\frac{\alpha^2}{2} + \sqrt{\frac{\alpha^4}{4} + k^4}}.$$

Consider the case when both ends of a beam are simply supported. The conditions at the left end $X = 0$, $X'' = 0$ give $C_2 = C_4 = 0$. Satisfying the same conditions at the right end, we obtain

$$C_1 \sinh s_1 l + C_3 \sin s_2 l = 0,$$

$$C_1 s_1^2 \sin s_1 l - C_3 s_2^2 \sin s_2 l = 0.$$

Equating the determinant of the coefficients of C_1 and C_3 to zero, we arrive at the equation

$$(s_1^2 + s_2^2) \sinh s_1 l \sin s_2 l = 0$$

or

$$\sin s_2 l = 0; \tag{8.41}$$

the roots of this frequency equation are

$$s_2 l = n\pi \quad (n = 1,\ 2,\ \ldots).$$

Consequently, the natural frequency is determined from the equation

$$l \sqrt{\frac{\alpha^2}{2} + \sqrt{\frac{\alpha^4}{4} + \frac{mp^2}{EI}}} = n\pi.$$

Taking into account expression (8.40), we find

$$p_n = \frac{n^2\pi^2}{l^2} \sqrt{\frac{EI}{m}} \sqrt{1 - \frac{Nl^2}{n^2\pi^2 EI}}\ . \tag{8.42}$$

The frequency increases in tension ($N < 0$) and decreases in compression. As the compressive force N approaches the critical value, the root tends to zero; at the same time the frequency p_n also tends to zero.

Effect of hoop forces. In the preceding discussion the longitudinal force was assumed to be given and independent of the displacements of the system. In some practical problems the longitudinal force accompanying the process of transverse vibration is induced by the bending of the beam and is a type of *reaction*.

Consider, for example, a beam with immovable hinge supports (see Fig. 68). When the beam is subjected to bending, the supports exert horizontal reactions which stretch the beam; the corresponding tensile force is called a *hoop force*. If the beam executes transverse vibrations, the hoop force varies in time.

If at time t the deflections of the beam are defined by the function $y(x)$, the elongation of the axis can be found from the formula

$$\Delta l = \frac{1}{2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx.$$

The corresponding hoop force is given by Hooke's law as

$$N = \frac{EA}{l}\Delta l = \frac{EA}{2l}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx.$$

Substitute this result in Eq. (8.38) for the longitudinal force N (with regard to sign)

$$\frac{\partial^2 y}{\partial t^2} - \frac{EA}{2ml}\frac{\partial^2 y}{\partial x^2}\int_0^l \left(\frac{\partial y}{\partial x}\right)^2 dx + \frac{EI}{m}\frac{\partial^4 y}{\partial x^4} = 0. \tag{8.43}$$

The above non-linear *integro-differential equation* is simplified in this case by substitution

$$y = a_n T_n \sin\frac{n\pi x}{l}, \tag{8.44}$$

where $T_n = T_n(t)$ is a dimensionless function of time whose maximum value may be put equal to any number (for example, unity); a_n is the amplitude of vibration.

Substituting expression (8.44) in Eq. (8.43), we obtain the *ordinary differential equation*

$$\ddot{T}_n + c_n T + d_n T^3 = 0 \tag{8.45}$$

in which the coefficients have the following values:

$$c_n = EAa_n^2\frac{n^4\pi^4}{4ml^4}, \quad d_n = EI\frac{n^4\pi^4}{ml^4}.$$

It is important to note that the differential equation (8.45) is non-linear; consequently, the frequency of free vibration depends on the amplitude (see Sec. 6).

The exact solution for the nth frequency of transverse vibration p_n^* is

$$p_n^* = p_n \varkappa,$$

where p_n is the frequency of transverse vibration for no hoop forces; $\varkappa$ is the correction factor depending on the ratio of the amplitude of vibration a_n to the radius of gyration ρ of the cross section.

For $n = 1$ the coefficient $\varkappa$ has the following values:

$a_1/2\rho$	0	0.5	1	2	5	10	100
$\varkappa$	1	1.09	1.31	1.98	4.75	8.53	84.72

When the amplitude and the radius of gyration of the cross section are comparable, the correction to the frequency becomes significant. If, for example, the amplitude of vibration of a beam of circular section is equal to its diameter, then $a_1/2\rho = 2$ and the frequency is almost twice as high as in the case of free displacement of the supports.

The case $a_1/2\rho \to \infty$ corresponds to the zero value of the radius of gyration when the flexural rigidity of the beam is vanishingly small (string). The formula for p^* then gives indeterminacy since $p = 0$ and $\varkappa \to \infty$. Opening this indeterminacy, we obtain the formula for the frequency of vibration of a string

$$p^* = \frac{4.181a}{l^2} \sqrt{\frac{EA}{m}}\,.$$

This formula applies to the case when the tension in the string is zero at the equilibrium position. Quite often the vibrating-string problem is stated *under other assumptions*; it is assumed that the displacements are small, the tensile force is specified and remains unchanged during vibration. The frequency formula is then

$$p^* = \frac{\pi}{l} \sqrt{\frac{N}{m}}\,,$$

where N is the constant tensile force.

Effect of Inelastic Resisting Forces

It was assumed above that the bar material was perfectly elastic and there was no friction. Let us consider the effect of internal friction assuming this to be viscous damping; the stress-strain relation is then given by

$$\sigma = E\varepsilon + c\,\frac{d\varepsilon}{dt}\,. \tag{8.46}$$

Consider the case of free *longitudinal* vibrations. In this case the longitudinal force is written as

$$N = \sigma F = EA\varepsilon + cA\,\frac{\partial\varepsilon}{\partial t} = EA\,\frac{\partial u}{\partial x} + cA\,\frac{\partial^2 u}{\partial x\,\partial t}\,. \tag{8.47}$$

From the equation of motion of an element of the bar there was obtained relation (8.2). Substituting in it expression (8.47), we arrive at the basic differential equation

$$k^2\,\frac{\partial^2 u}{\partial x^2} + \frac{c}{\rho}\,\frac{\partial^3 u}{\partial x^2\,\partial t} = \frac{\partial^2 u}{\partial t^2} \tag{8.48}$$

which differs from Eq. (8.3) by the second term which takes account of the effect of damping forces.

Following Fourier's method, we seek a solution of Eq. (8.48) in the form

$$u = \sum_{r=1}^{\infty} X_r(x) T_r(t), \tag{8.49}$$

where X_r is a function of the co-ordinate x only, and T_r is a function of the time t only.

Each term of the series must satisfy the boundary conditions of the problem and the entire sum must also satisfy the initial conditions. Substituting expression (8.49) in Eq. (8.48) and requiring that the equality be satisfied for every number r, we obtain

$$k^2 X_r'' T_r + \frac{c}{\rho} X_r'' \dot{T}_r = X_r \ddot{T}_r; \tag{8.50}$$

here and hereafter primes denote differentiation with respect to the co-ordinate x and dots with respect to time t.

Dividing equality (8.50) by the product $X_r \left(T_r + \frac{c}{E} T_r\right)$, we arrive at the equality

$$k^2 \frac{X_r''}{X} = \frac{\ddot{T}_r}{T_r + \frac{c}{E} \dot{T}_r} \tag{8.51}$$

the left-hand side of which may depend only on the co-ordinate x and the right-hand side only on the time t. Since equality (8.51) must be satisfied identically, both sides must be equal to one and the same constant, which will be denoted by $-p_r^2$.

Hence we have the following equations

$$X_r'' + \frac{p_r^2}{k^2} X_r = 0, \tag{8.52}$$

$$\ddot{T}_r + \frac{c}{E} p_r^2 \dot{T}_r + p_r^2 T_r = 0. \tag{8.53}$$

Equation (8.52) is entirely independent of the damping constant c and, in particular, remains the same in the case of a perfectly elastic system when $c = 0$. Therefore, the numbers p_r are identical with those found above; however, as will be shown later, the quantity p_r gives only an approximate value for the natural frequency. It is important to note that the natural modes are entirely independent of the damping properties of the bar, i.e., *the modes of free damped vibration are identical with the modes of free undamped vibration.*

Revert now to Eq. (8.53) which governs obviously the process of damped vibration; its solution is

$$T_r = e^{-n_r t}(A_r \sin p_r^* t + B_r \cos p_r^* t). \tag{8.54}$$

Here

$$n_r = \frac{cp_r^2}{2E}, \tag{8.55}$$

$$p_r^* = \sqrt{p_r^2 - n_r^2} = p_r \sqrt{1 - \left(\frac{cp_r^2}{2E}\right)^2}. \tag{8.56}$$

Expression (8.55) defines the rate of damping and expression (8.56) the *frequency of vibration.*

Thus, the complete solution of the equation of the problem is

$$u = \sum_{r=1}^{\infty} X_r e^{-n_r t}(A_r \sin p_r^* t + B_r \cos p_r^* t). \tag{8.57}$$

The constants C_r and D_r can always be found from the given initial conditions. Let the initial displacements and the initial velocities of all sections of the bar be assigned as follows:

$$u(x, 0) = f_1(x), \qquad \frac{\partial u}{\partial t}(x, 0) = f_2(x), \tag{8.58}$$

where $f_1(x)$ and $f_2(x)$ are known functions.

According to Eqs. (8.57) and (8.58) we have at $t = 0$

$$f_1(x) = \sum_{r=1}^{\infty} A_r X_r(x),$$

$$f_2(x) = \sum_{r=1}^{\infty} (A_r p_r^* - B_r n_r) X_r(x).$$

Multiplying both sides of these equalities by $X_r(x)$ and integrating over the total length of the bar, we obtain

$$\left.\begin{aligned} \int_0^l f_1(x) X_r(x)\,dx &= B_r \int_0^l [X_r(x)]^2\,dx, \\ \int_0^l f_2(x) X_r(x)\,dx &= (A_r p_r^* - B_r n_r) \int_0^l [X_r(x)]^2\,dx. \end{aligned}\right\} \tag{8.59}$$

According to the orthogonality condition for the natural modes, all the other terms on the right-hand side of these equalities vanish. It is easy now to find A_r and B_r from equality (8.59) for any number r.

Examining expressions (8.55) and (8.57), we note that the higher the number of the mode of vibration X_r, the more rapid is the damping. Moreover, the terms in expression (8.57) describe a damped vibration if p_r^* is a real number. From expression (8.56) it is seen that this occurs only for several initial values of r as long as the following inequality is fulfilled

$$\frac{cp_r^2}{2E} < 1. \tag{8.60}$$

For sufficiently large values of r inequality (8.60) is violated and the quantity p_r^* becomes imaginary. The corresponding terms of the general solution (8.57) will no longer describe a damped vibration but will represent an aperiodic damped motion. In other words, the vibration in the true sense of the word is expressed by only a *finite part of sum* (8.57).

All these qualitative conclusions are true not only for longitudinal vibrations but also for torsional and flexural vibrations.

9. Vibrations of Bars of Variable Section

Basic Equations

In cases where the distributed mass and the section of a bar are not uniform along its length, the equation of longitudinal vibrations (8.3) should be replaced by the equation

$$k^2 \frac{\partial}{\partial x}\left(A \frac{\partial u}{\partial x}\right) = A \frac{\partial^2 u}{\partial t^2}, \tag{9.1}$$

the equation of torsional vibrations (8.14) by

$$k_1^2 \frac{\partial}{\partial x}\left(J \frac{\partial \varphi}{\partial x}\right) = J \frac{\partial^2 \varphi}{\partial t^2} \tag{9.2}$$

and the equation of transverse vibrations (8.19) by

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 y}{\partial x^2}\right) = -m \frac{\partial^2 y}{\partial t^2}. \tag{9.3}$$

Equations (9.1) to (9.3) can be reduced to ordinary differential equations for the function $X(x)$ by means of identical substitutions $u = X(x)T(t)$, $\varphi = X(x)T(t)$, $y = X(x)T(t)$

$$(AX')' + \frac{p^2}{k^2} AX = 0, \tag{9.4}$$

$$(JX')' + \frac{p^2}{k_1^2} JX = 0, \tag{9.5}$$

$$(EIX'')'' - mp^2X = 0 \tag{9.6}$$

and to one equation for the function $T(t)$.

Equations (9.4) to (9.6) have *variable coefficients* in contrast to the equations solved above.

The closed form of solutions can be obtained only in particular cases when the variables A, J, EI, m are defined by special relations. In the general case recourse must of necessity be made to approximate methods. Thus, one possible approach is to concentrate the distributed mass in a number of points along the length of the bar, in which case the system retains only a finite number of degrees of freedom equal to the number of points of reduction. Extensive use is made of various modifications of the variational method (see below). Finally, the method of successive approximations may be used to great advantage.

Rayleigh's Theorem

According to this theorem, the true value of the lowest natural frequency is always less than the approximate value of the frequency found by the energy method. We shall prove this theorem for the case of flexural vibrations; it can be proved for other cases in a similar way.

Assume that, in solving the problem of free flexural vibration by the energy method, the mode of vibration is chosen as $f = f(x)$. The corresponding static loading [i.e., the loading capable of producing deflections described by the curve $f(x)$] can then be represented as

$$q = (EIf'')''$$

and hence the approximate expression for the square of the frequency (4.26) is

$$p^2 = \frac{\int\limits_0^l (EIf'')'' f\,dx}{\int\limits_0^l mf^2\,dx}. \tag{9.7}$$

Because of the arbitrary choice of the function $f(x)$ it does not coincide with any of the natural modes which give exact solutions; however, the function $f(x)$ can be represented as a series in terms of these modes. If the lowest natural frequency is sought, the function $f(x)$ may be represented as

$$f(x) = X_1(x) + b_2X_2(x) + b_3X_3(x) + \dots . \tag{9.8}$$

If a good guess is made for the mode of vibration, the function $f(x)$ is close to $X_1(x)$ and hence the coefficients b_2, b_3, ... are small numbers.

Differentiate expression (9.8) twice with respect to x, next multiply both sides by the flexural rigidity EI and again differentiate the result twice. We then obtain

$$(EIf'')'' = (EIX_1'')'' + b_2(EIX_2'')'' + b_3(EIX_3'')'' + \dots . \tag{9.9}$$

According to the basic equation (9.6) we can write

$$(EIX_1'')'' = mp_1^2X, \qquad (EIX_2'')'' = mp_2^2X, \ \ldots \ .$$

Substituting in expression (9.9), we obtain

$$(EIf'')'' = m\,(p_1^2X_1 + b_2p_2^2X_2 + b_3p_3^2X_3 + \ldots). \tag{9.10}$$

Using expressions (9.8) and (9.10) we form the numerator of formula (9.7)

$$\int_0^l (EIf'')''\, f\, dx =$$

$$= \int_0^l m\,(X_1 + b_2X_2 + b_3X_3 + \ldots)\,(p_1^2X_1 + b_2p_2^2X_2 + b_3p_3^2X_3 + \ldots)\, dx.$$

Due to the orthogonality of the natural modes, all the integrals of products where the subscripts of the multipliers are different are equal to zero; therefore

$$\int_0^l (EIf'')''f\,dx = p_1^2\int_0^l mX_1^2\,dx + p_2^2b_2^2\int_0^l mX_2^2\,dx + \ldots \ . \tag{9.11}$$

The denominator of formula (9.7) is obtained by using expression (9.8)

$$\int_0^l mf^2\,dx = \int_0^l m\,(X_1 + b_2X_2 + b_3X_3 + \ldots)^2\,dx =$$

$$= \int_0^l mX_1^2\,dx + b_2^2\int_0^l mX_2^2\,dx + b_3^2\int_0^l mX_3^2 + \ldots \ . \tag{9.12}$$

Here again, all the terms containing the products X_mX_n vanish. Substituting expressions (9.11) and (9.12) in formula (9.7), we obtain the square of the lowest frequency

$$p^2 = \frac{p_1^2\int_0^l mX_1^2\,dx + p_2^2b_2^2\int_0^l mX_2^2\,dx + \ldots}{\int_0^l mX_1^2\,dx + b_2^2\int_0^l mX_2^2\,dx + \ldots} =$$

$$= p_1^2\,\frac{\int_0^l mX_1^2\,dx + \frac{p_2^2}{p_1^2}\,b_2^2\int_0^l mX_2^2\,dx + \ldots}{\int_0^l mX_1^2\,dx + b_2^2\int_0^l mX_2^2\,dx + \ldots}\,. \tag{9.13}$$

Since $p_1 < p_2 < p_3, \ldots$, all fractions p_i^2/p_1^2 are greater than unity and hence all terms of the numerator, beginning with the second one, are greater than the corresponding terms of the denominator. Consequently, the whole fraction appearing in expression (9.13) is greater than unity, i.e.,

$$p^2 > p_1^2, \tag{9.14}$$

as stated by Rayleigh's theorem.

Inequality (9.14) is valid not only for flexural but also for longitudinal and torsional vibrations.

Ritz's Method

Assign several functions, $f_1(x), f_2(x), \ldots, f_n(x)$ each of which satisfies the geometric boundary conditions of the problem, and form the function $f(x)$ as the sum

$$f(x) = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n. \tag{9.15}$$

If this function is substituted in Rayleigh's formula (4.26)

$$p^2 = \frac{\int\limits_0^l EI(f'')^2\,dx}{\int\limits_0^l mf^2\,dx}, \tag{9.16}$$

the result will depend on the particular choice of coefficients $c_1, c_2, \ldots, c_n$.

Ritz's method can be developed using a simple idea: the coefficients $c_1, c_2, \ldots, c_n$ must be chosen so that formula (9.16) will give the lowest value for p^2. From Rayleigh's theorem it follows that this choice will provide the best approximation (for a given set of functions f_i).*

The minimum conditions for p^2 are

$$\frac{\partial}{\partial c_i}\,\frac{\int\limits_0^l EI(f'')^2\,dx}{\int\limits_0^l mf^2\,dx} = 0 \qquad (i = 1, 2, \ldots),$$

i.e.,

$$\left[\frac{\partial}{\partial c_i}\int\limits_0^l EI(f'')^2\,dx\right]\left[\int\limits_0^l mf^2\,dx\right] - \left[\frac{\partial}{\partial c_i}\int\limits_0^l mf^2\,dx\right]\left[\int\limits_0^l EI(f'')^2\,dx\right] = 0.$$

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* Walther Ritz himself used other considerations.

Dividing this equation by the integral $\int\limits_0^l mf^2\,dx$ and taking into account formula (9.16), we obtain

$$\frac{\partial}{\partial c_i}\int\limits_0^l [EI\,(f'')^2-p^2mf^2]\,dx=0 \qquad (i=1,\,2,\,\ldots). \qquad (9.17)$$

Equations (9.17) are homogeneous and linear in c_1, c_2, . . . and their number is equal to the number of terms in expression (9.15). Equating the determinant of the coefficients of c_1, c_2, . . . to zero, we obtain the frequency equation. This equation gives not only a good approximation to the lowest frequency but also the values of higher frequencies (though to a lesser degree of accuracy); it is then possible to find as many frequencies as there are terms in expression (9.15).

Ritz's method, just as Rayleigh's method, provides a means for solving the problem when the functions EI and m are discontinuous or when these functions are represented by different analytic expressions in different portions of the beam.

Sometimes the same idea is used in an alternate form. In the analysis of transverse vibrations of turbine blades, for instance, a function $f(x) = ax^s$ is assigned (the origin of co-ordinates is at the fixed end). By applying Rayleigh's formula (9.16), the frequency is obtained as a function of the exponent s. Then the value of s is determined (usually by calculation) to which the lowest frequency corresponds. This allows both the mode and frequency of vibration of the first tone to be determined in a rather reliable way.

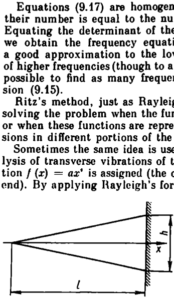

Fig. 69

Example 15. Using Ritz's method, determine the lowest natural frequency of transverse vibration of a cantilever beam of variable section having a width of unity; the depth varies linearly: $h_x = \frac{x}{l}h$ (Fig. 69); $I = \frac{h^3}{12l^3}x^3$; $m = \frac{\rho h}{l}x$.

The exact value of the lowest frequency as found by Kirchhoff is

$$p=\frac{2.657h}{l^2}\sqrt{\frac{E}{3\rho}}\,.$$

For approximate solution we assume

$$f(x)=c_1f_1(x)+c_2f_2(x)+\ldots=c_1\left(1-\frac{x}{l}\right)^2+c^2\frac{x}{l}\left(1-\frac{x}{l}\right)^2+\ldots; \qquad (9.18)$$

each term of this expansion satisfies the boundary conditions of the problem: $f_i(x)=0,\ f_i'(x)=0$ at $x=l$.

Limiting the series to one term, we obtain by Rayleigh's method (the error is about 3 per cent)

$$p=\frac{2.740h}{l^2}\sqrt{\frac{E}{3\rho}}\,.$$

In order to obtain a closer approximation, we take two terms of expansion (9.18) and substitute them in expression (9.17)

$$\frac{h^3}{12l^3}\left[{}^{1}(c_1-2c_2)^2+\frac{24}{5}c_2(c_1-2c_2)+6c_2^2\right]-p^2\frac{\rho hl}{E}\left(\frac{c_1^2}{30}+\frac{2c_1c_2}{105}+\frac{c_2^2}{280}\right).$$

Differentiating this expression with respect to c_1 and c_2 successively, we arrive at the equations

$$\left.\begin{aligned}\left(\frac{Eh^2}{12\rho l^4}-\frac{p^2}{30}\right)c_1+\left(\frac{Eh^2}{30\rho l^4}-\frac{p^2}{105}\right)c_2=0,\\ \left(\frac{Eh^2}{30\rho l^4}-\frac{p^2}{105}\right)c_1+\left(\frac{Eh^2}{30\rho l^4}-\frac{p^2}{280}\right)c_2=0.\end{aligned}\right\}\tag{9.19}$$

Equating the determinant of the coefficients of these equations to zero, we obtain the frequency equation; the smallest root of the equation

$$p=\frac{2.660h}{l^2}\sqrt{\frac{E}{3\rho}}$$

gives an error of 0.1 per cent as against the exact value.

Bubnov-Galerkin Method

In this method, it is necessary to substitute an approximately chosen expression $f(x)$ for $X(x)$ in Eq. (9.6) and to form the integral

$$\int_0^l [(EIf'')''-mp^2f]\,f\,dx=0;\tag{9.20}$$

from this Rayleigh's formula (9.7) can be obtained as a special case.

Assuming $f(x)$ in the form of (9.15) and regarding each of the terms as a virtual displacement, we obtain, in place of (9.20), a relation expressing that the virtual work is equal to zero

$$\int_0^l [(EIf'')''-mp^2f]\,f_i\,dx=0.\tag{9.21}$$

We can write as many such equalities as there are terms in the assumed expression (9.15). Each of Eqs. (9.21) is homogeneous and is of the first degree in the undetermined quantities $c_1, c_2, \ldots$. Equating the determinant of system (9.21) to zero, we obtain the frequency equation.

The Bubnov-Galerkin method has a peculiar feature relating to the boundary conditions. If the functions $f_i(x)$ satisfy only the *geometric boundary conditions*, serious errors may result in the solution by the Bubnov-Galerkin method. The point is that by ignoring the *force boundary conditions* for the functions $f_i(x)$ (for instance, by disregarding the condition $f_i'' = 0$ at a simply supported end or the conditions $f_i'' = 0$ and $f_i''' = 0$ at a free end) we implicitly assume the existence of such *boundary conditions* which are actually absent. Evaluating then the total work, we introduce therewith the work of these forces, this being the source of the error. To offset the error it is necessary to subtract the "excess" work of these boundary forces from the left-hand side of expression (9.21). In this case Ritz's and Bubnov-Galerkin methods give the same results.

A common procedure, however, is to subject the functions beforehand not only to geometric but also to force boundary conditions.

Example 16. Using the Bubnov-Galerkin method determine the lowest frequency of transverse vibration of the cantilever beam considered in Example 15 (Fig. 69).

Assuming the mode of vibration as

$$f(x) = c_1\left(1-\frac{x}{l}\right)^2 + c_2\frac{x}{l}\left(1-\frac{x}{l}\right)^2,$$

we satisfy both the geometric conditions at the right end and the force conditions at the left end.

Differentiating $f(x)$ twice, multiplying by

$$EI = \frac{Eh^3}{12l^3}x^3$$

and again differentiating twice, we obtain

$$(EIf'')'' = \frac{Eh^3}{l^5}\left[(c_1-2c_2)x + \frac{6c_2x^2}{l}\right].$$

Substituting in expression (9.21), we find

$$\int_0^l \left\{\left[\frac{Eh^3}{l^5}(c_1-2c_2)x + \frac{6c_2x^2}{l}\right] - \right.$$

$$\left. -\frac{\rho h p^2}{l}\left[c_1x\left(1-\frac{x}{l}\right)^2 + c_2\frac{x}{l}\left(1-\frac{x}{l}\right)^2\right]\right\}\left(1-\frac{x}{l}\right)^2 dx = 0,$$

$$\int_0^l \left\{\frac{Eh^3}{l^5}\left[(c_1-2c_2)x + \frac{6c_2x^2}{l}\right] - \right.$$

$$\left. -\frac{\rho h p^2}{l}\left[c_1x\left(1-\frac{x}{l}\right)^2 + c_2\frac{x^2}{l}\left(1-\frac{x}{l}\right)\right]\right\}\frac{x}{l}\left(1-\frac{x}{l}\right)^2 dx = 0.$$

Hence we obtain the same equations (9.19) as by Ritz's method.

In problems involving longitudinal and torsional vibrations of bars, the equations of the Bubnov-Galerkin method are set up in a similar way.

Method of Successive Approximations

We shall first prove that the usual process of successive approximations (see Sec. 7) leads to the first natural mode. The process is based on the comparison of two curves, a_n and a_{n+1}, the latter being obtained as the deflection curve due to the load ma_n; the approximate value of the square of the frequency is then given by the formula

$$p_1^2 = \frac{(a_{n+1})_{\max}}{(a_n)_{\max}}. \tag{9.22}$$

Similarly to expression (9.8) we represent the starting curve $a_1(x)$ as a series

$$a_1(x) = X_1(x) + b_2X_2(x) + b_3X_3(x) + \dots . \tag{9.23}$$

The load corresponding to the deflections a_1 is then

$$ma_1 = mX_1 + mb_2X_2 + mb_3X_3 + \dots . \tag{9.24}$$

Consider one of the components of this load, mb_iX_i. The deflections due to the load $mp_i^2X_i$ are X_i; hence the deflections due to the load mb_iX_i are b_i/p_i^2 times as large, i.e., are equal to $\frac{b_i}{p_i^2}X_i$. Consequently, the deflection curve due to the total load is defined by the series

$$a_2 = \frac{X_1}{p_1^2} + \frac{b_2X_2}{p_2^2} + \frac{b_3X_3}{p_3^2} + \dots \tag{9.25}$$

which differs from series (9.24) in that each term is divided by the square of the corresponding frequency. Since $p_1 < p_2$, $p_2 < p_3$, ..., the curve $a_2(x)$ is closer to $X_1(x)$ than the starting curve $a_1(x)$; the terms which contain $X_2(x)$, $X_3(x)$, ... and distort the fundamental mode $X_1(x)$ are presented in series (9.25) to a lesser degree than in series (9.23). Carrying on the process, we obtain for the $(n+1)$th curve

$$a_{n+1} = \frac{1}{p_1^{2n}}\left[X_1 + b_2\left(\frac{p_1}{p_2}\right)^{2n}X_2 + b_3\left(\frac{p_1}{p_3}\right)^{2n}X_3 + \dots\right]. \tag{9.26}$$

As can be seen, the higher modes vanish as $n \to \infty$ and consequently, no matter what the starting curve may be (for instance, it may even be very close to the second natural mode), the process will eventually lead precisely to the first natural mode.

It would appear, therefore, that the attempt to construct the second natural mode by this method will fail as any distortion introduced by the first mode into the approximate second mode will increase gradually and, after a large number of constructions, the second type of vibration will vanish altogether and only the first type will remain.

However, by modifying somewhat the method we can have not the first but precisely the second natural mode of vibration "refined" as a result of successive approximations; this procedure has found application in the analysis of flexural vibrations of airplane wings and turbine blades.

The procedure is based on the elimination of the mode $X_1(x)$ from the starting function $a_1(x)$. Suppose that the term corresponding to the first mode is actually absent in expansion (9.23); it cannot then appear in any of the subsequent operations and series (9.26) becomes

$$a_{n+1}=\frac{1}{p_2^{2n}}\left[X_2+b_3\left(\frac{p_2}{p_3}\right)^{2n}X_3+b_4\left(\frac{p_2}{p_4}\right)^{2n}X_4+\dots\right].$$

As $n \to \infty$ all the modes of vibration but *the second* will vanish. In order that the process of successive approximations may lead precisely to the second mode, it is necessary to eliminate the first natural mode $X_1(x)$ from the starting function $a_1(x)$.

This can be done by using, as a basis for constructing the second approximation, the function

$$\bar{a}_1(x)=a_1(x)-\alpha_1 X_1(x), \tag{9.27}$$

where $a_1(x)$ = suitable function,
$X_1(x)$ = predetermined first natural mode.

The coefficient α should be chosen so as to make the mode $\bar{a}_1(x)$ orthogonal to the first natural mode $X_1(x)$

$$\int_0^l m\bar{a}_1(x)\,X_1(x)\,dx=0.$$

Substituting expression (9.27), we obtain

$$\alpha_1=\frac{\int_0^l ma_1(x)\,X_1(x)\,dx}{\int_0^l mX_1^2(x)\,dx}.$$

Further, the deflections a_2 due to the load $m\bar{a}_1$ must be determined. If the first mode X_1 is eliminated altogether by the use of expression (9.27), the function a_2 will be closer to the second mode and the subsequent operations will provide an arbitrarily close approximation to X_2.

However, the first natural mode X_1 can be known only approximately; therefore, in practice the operation indicated in expression

(9.27) does not ensure the complete elimination of the first mode X_1. This makes it necessary to correct the function a_2 again in carrying on the process and to take

$$\bar{a}_2(x) = a_2(x) - \alpha_2 X_1(x), \tag{9.28}$$

where the coefficient α_2 is also determined by the condition of orthogonality of the functions $\bar{a}_2$ and X_1

$$\int_0^l m\bar{a}_2(x)\, X_1(x)\, dx = 0$$

which, after substitution of expression (9.28), yields

$$\alpha_2 = \frac{\int_0^l ma_2(x)\, X_1(x)\, dx}{\int_0^l mX_1^2(x)\, dx}.$$

Next, the curve a_3 due to the load $m\bar{a}_2$ must be determined and corrected again by the formula

$$\bar{a}_3(x) = a_3(x) - \alpha_3 X_1(x)$$

and so on.

In this process of successive approximations the orthogonalization accompanies every step of the computations and, continuously deleting the "traces" of the first mode, leads to the second natural mode and the second frequency; the latter is defined, similarly to expression (9.22), by the formula

$$p_2^2 = \frac{(\bar{a}_{n+1})_{\max}}{(\bar{a}_n)_{\max}}.$$

In the same way, the accompanying orthogonalization enables us to determine the third normal mode and the third frequency, etc.

10. Two-Dimensional Vibrations of Disks

In the analysis of vibrations of rotating disks it is customary to assume that a disk is a body of revolution having a plane of symmetry perpendicular to the axis of rotation ("middle plane"); it is further assumed that the inclination of the lateral surfaces of the disk to this plane is very small, and the thickness of the disk is small compared to its diameter.

Under these assumptions we may speak of two types of vibration. The first type of vibration comprises cases where any point in the middle plane of a disk vibrates in the same plane, i.e., executes *two-dimensional vibrations*; these in turn may be subdivided into *radial* and *tangential* vibrations. Vibrations of the second type are flexural vibrations of a disk which are characterized by a three-dimensional picture of the deformation and by displacements of points of the middle plane along a perpendicular to this plane. It has been established that the modes and frequencies of free two-dimensional vibration are virtually unaffected by the centrifugal effects due to the rotation of the disk; therefore, the rotation of the disk is taken into consideration only in the analysis of flexural vibrations.

Radial Vibrations

Radial vibrations are axisymmetric two-dimensional vibrations in which any point of the middle plane of a disk moves only along the corresponding radius; there are no displacements in a circumferential (tangential) direction or in a direction perpendicular to the middle plane of the disk.

Imagine an infinitesimal element of the disk to be isolated by two radial and two cylindrical sections (Fig. 70). Denote by r the distance from the centre of the disk to this element and by $d\theta$ the corresponding infinitesimal central angle. The normal stresses on the circumferential and radial sections are denoted by σ_r and σ_θ, respectively; because of the axial symmetry no shearing stresses occur on the faces of the isolated element. The normal forces on both lateral faces of the element are $\sigma_\theta h\,dr$, where h is the thickness of the disk which is in general a function of the co-ordinate r. The normal force on the inner face is equal to the product of the stress σ_r and the area of the face $hr\,d\theta$, i.e.., it is equal to $\sigma_r hr\,d\theta$. The normal force on the outer face receives an increment $\frac{\partial}{\partial r}(\sigma_r hr\,d\theta)\,dr$ and thus is equal to $\sigma_r hr\,d\theta+\frac{\partial}{\partial r}(\sigma_r hr\,d\theta)\,dr$.

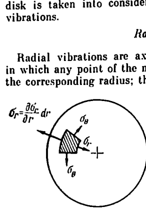

Fig. 70

Based on Hooke's law for the state of plane stress, we can write

$$\varepsilon_r=\frac{\sigma_r}{E}-\mu\frac{\sigma_\theta}{E},$$

$$\varepsilon_\theta=\frac{\sigma_\theta}{E}-\mu\frac{\sigma_r}{E},$$

where μ is Poisson's ratio.

We invert the last relations to express stresses in terms of strains

$$\left.\begin{aligned}\sigma_r &= \frac{E}{1-\mu^2}(\varepsilon_r+\mu\varepsilon_\theta),\\ \sigma_\theta &= \frac{E}{1-\mu^2}(\varepsilon_\theta+\mu\varepsilon_r).\end{aligned}\right\} \qquad (10.1)$$

Recalling the well-known formulas of strength of materials

$$\varepsilon_r=\frac{\partial u}{\partial r}, \qquad \varepsilon_\theta=\frac{u}{r}$$

which express strains in terms of displacements (u is the radial displacement of the current point), we write relations (10.1) as

$$\left.\begin{aligned}\sigma_r &= \frac{E}{1-\mu^2}\left(\frac{\partial u}{\partial r}+\mu\frac{u}{r}\right),\\ \sigma_\theta &= \frac{E}{1-\mu^2}\left(\frac{u}{r}+\mu\frac{\partial u}{\partial r}\right).\end{aligned}\right\} \qquad (10.2)$$

Thus, both normal stresses, σ_r and σ_θ, are expressed in terms of one function, $u = u(r, t)$. By projecting all the forces acting on the element on the direction of the radius, we obtain the differential equation of motion

$$\left[\sigma_r hr+\frac{\partial}{\partial r}(\sigma_r hr)\,dr\right]d\theta-\sigma_r hr\,d\theta-2\sigma_\theta h\,dr\,\frac{d\theta}{2}=\rho hr\ddot{u}\,dr\,d\theta,$$

where ρ is the density of the material of the disk. The right-hand side of the differential equation is written taking into account that $\rho hr\,dr\,d\theta$ is the mass of the element.

Substituting expressions (10.2) and transforming, we arrive at the basic differential equation of the problem of free radial vibration of a disk

$$u''+u'\left(\frac{1}{r}+\frac{h'}{h}\right)+u\left(\frac{\mu h'}{rh}-\frac{1}{r^2}\right)=\frac{\rho\ddot{u}}{E}(1-\mu)^2. \qquad (10.3)$$

To solve this equation, we make use of Fourier's method and seek the solution as the product of two functions

$$u(r, t)=R(r)\,T(t), \qquad (10.4)$$

where $R(r)$ = function of radius r,
$T(t)$ = function of time t.

Substituting solution (10.4) in the basic equation (10.3), we separate the variables and obtain the differential equation for the unknown function $R(r)$ which, as may be inferred from the solution, defines the natural mode of vibration

$$R''+R'\left(\frac{1}{r}+\frac{h'}{h}\right)+R\left(\frac{\mu h'}{rh}-\frac{1}{r^2}+\frac{p^2}{k^2}\right)=0; \qquad (10.5)$$

here p is the unknown natural frequency,

$$k=\sqrt{\frac{E}{\rho(1-\mu^2)}}.$$

Since the coefficients of Eq. (10.5) are variable, the solution presents difficulties as a rule. Only in two particular cases can the equation be somewhat simplified and solved in terms of Bessel functions; these are the case of a two-dimensional disk when $h' = 0$

$$R''+\frac{R'}{r}+R\left(\frac{p^2}{k^2}-\frac{1}{r^2}\right)=0, \tag{10.6}$$

and the case of a disk of hyperbolic section when $h=\frac{k}{r^n}$

$$R''+R'\frac{1-n}{r}+R\left(\frac{p^2}{k^2}-\frac{1-\mu n}{r^2}\right)=0. \tag{10.7}$$

In the general case we can use the Bubnov-Galerkin method and, assigning the mode of vibration $R(r)$, determine the natural frequency p from the equation

$$\int_{r_1}^{r_2}\left[R''+R'\left(\frac{1}{r}+\frac{h'}{h}\right)+R\left(\frac{\mu h'}{rh}-\frac{1}{r^2}+\frac{p^2}{k^2}\right)\right]R\,dr=0,$$

where r_1 and r_2 are the inner and outer radii of the disk.

The function $R(r)$ must be chosen so as to satisfy the boundary conditions. A *fixed contour* involves no displacements u and consequently the condition $R = 0$ must be satisfied. A *free contour* involves no stresses σ_r and, according to expression (10.2), the boundary condition is

$$R'+\mu\frac{R}{r}=0.$$

Tangential Vibrations

The occurrence of tangential vibrations may be visualized as follows. Let a disk mounted on a fixed shaft be loaded at its outer contour with tangential forces forming a couple and at a certain instant the load be suddenly released. Then free tangential vibrations are set up in which the points of the disk located on a circle remain on this circle and all the radii distort identically.

Consider two infinitely close points, a and b, in the middle plane of the disk which lie on the same radius before deformation. Let their new positions after tangential deformation be defined by points a_1 and b_1 (Fig. 71). The radius vector of the point a rotates through an angle φ and occupies a position Oa_1 and the radius vector of the point b rotates through an angle $\varphi + d\varphi$ and occupies a position Ob_1.

The angle γ between the direction of the rotated element a_1b_1 and the new direction of the radius Oa_1b_2 is the angle of shear. From $\Delta a_1b_1b_2$ we have $b_1b_2 = \gamma\, dr$ and from ΔOb_1b_2

$$b_1b_2 = (r + dr)\, d\varphi.$$

Consequently, the shearing strain γ is expressed as

$$\gamma = r\frac{d\varphi}{dr}.$$

To this picture of the deformation correspond only the shearing stresses on the faces of an infinitesimal element (see Fig. 70)

$$\tau = Gr\frac{d\varphi}{dr}. \qquad (10.8)$$

The normal stresses on these faces are zero. Therefore, on the inner face the tangential force is $\tau hr\, d\theta$ and on the outer face $\tau hr\, d\theta + = \frac{\partial(\tau hr)}{\partial r}dr\, d\theta$. The lateral (radial) faces are also acted on by tangential forces but their expressions are not needed in this case as we shall set up the moment equation with respect to the centre of the disk. This equation is

$$\tau hr^2\, d\theta - \left[\tau hr + \frac{\partial(\tau hr)}{\partial r}dr\right](r + dr)\, d\theta =$$
$$= \rho hr^2\, dr\, d\theta r\frac{\partial^2\varphi}{\partial t^2}.$$

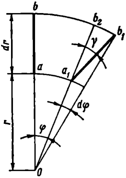

Fig. 71

The right-hand side represents the moment of the inertia force equal to the mass of the element $\rho hr dr d\theta$ times the tangential acceleration $\partial^2\varphi/\partial t^2$. Replacing the shearing stress by expression (10.8), we obtain, after transformation, the differential equation for the angle of rotation φ

$$\varphi'' + \varphi'\left(\frac{3}{r} + \frac{h'}{h}\right) + \frac{\rho}{G}\ddot{\varphi} = 0.$$

As before, by substitution

$$\varphi(r, t) = R(r)\, T(t)$$

this partial differential equation is reduced to the ordinary differential equation

$$R'' + R'\left(\frac{3}{r} + \frac{h'}{h}\right) - R\frac{p^2}{k_1^2} = 0, \qquad (10.9)$$

where p is the natural frequency of vibration,

$$k_1=\sqrt{\frac{G}{\rho}}\,.$$

The equation obtained is of the same type as Eq. (10.5) and in the general case of the relation $h = h(r)$ we have to resort to approximate methods of solution.

In choosing the function $R(r)$ it is necessary to ensure the fulfilment of the boundary conditions which are as follows:

At a fixed contour, $\varphi = 0$, consequently

$$R=0. \tag{10.10}$$

At a free contour, $\tau=0$, i.e., according to formula (10.8)

$$R'=0. \tag{10.11}$$

If the contour *carries a mass* of intensity m *distributed along the perimeter*, the inertia force developed by this mass (per unit length of the contour) is $-mr\ddot{\varphi}$; it must be equated to the tangential force $\tau h = Grh\varphi'$ also estimated per unit length of the contour: $-mr\ddot{\varphi} = Grh\varphi'$, i.e., $-mrR\ddot{T} = GrhR'T$. But since $\ddot{T} = -p^2T$, the boundary condition is $mp^2R = GhR'$. In choosing the signs here, we assumed that the distributed mass was carried by the outer contour of the disk.

After the function $R(r)$ is chosen, the natural frequency can be found by the Bubnov-Galerkin method from the expression

$$\int_{r_1}^{r_2}\left[R''+R'\left(\frac{3}{r}+\frac{h'}{h}\right)+R\frac{p^2}{k_1^2}\right]R\,dr=0.$$

In one particular case there is no need to use approximate methods since it is easy to find the corresponding exact solution. This is the case of a disk of hyperbolic section for which $h = b/r^3$. Indeed, $\frac{h'}{h}=-\frac{3}{r}$ and Eq. (10.9) becomes

$$R''+R\frac{p^2}{k_1^2}=0.$$

The solution of this equation

$$R=A\sin\frac{pr}{k_1}+B\cos\frac{pr}{k_1}$$

must be subjected to boundary conditions. Let, for instance, the inner contour of the disk be fixed and the outer contour free. Then, according to expressions (10.10) and (10.11), we have

$$A\sin\frac{pr_1}{k_1}+B\cos\frac{pr_1}{k_1}=0,$$

$$\frac{Ap}{k_1}\cos\frac{pr_2}{k_1}-\frac{Bp}{k_1}\sin\frac{pr_1}{k_1}=0.$$

Equating the determinant of this system to zero, we obtain the frequency equation

$$\sin\frac{pr_1}{k_1}\sin\frac{pr_2}{k_1}+\cos\frac{pr_1}{k_1}\cos\frac{pr_2}{k_1}=0$$

or

$$\cos\frac{p(r_2-r_1)}{k_1}=0;$$

the smallest root of this equation is

$$\frac{p(r_2-r_1)}{k_1}=\frac{\pi}{2};$$

hence we find the first natural frequency of tangential vibration

$$p=\frac{\pi}{2}\frac{k_1}{r_2-r_1}.$$

11. Flexural Vibrations of Disks

General Relationships

Flexural vibrations may be divided into two types:

(a) Axisymmetric *umbrella vibrations* in which all the radii of the disk bend identically out of the plane of the disk. This mode of vibration is shown in Fig. 72*a* and *b*, where the plus and minus signs indicate the direction of displacement of the corresponding circular zones of the disk. The middle plane deforms into a surface

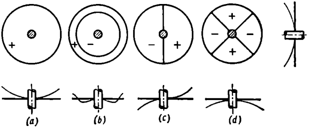

Fig. 72

of revolution during vibration; there may be one or more *fixed circles*, depending on the number of the normal mode of vibration.

(b) *Fan vibrations* which are characterized by the existence of one or more *nodal diameters* (Fig. 72*c* and *d*).

In the general cases the corresponding differential equations have variable coefficients and do not yield to integration in terms of elementary functions even in simple cases. These problems are usually solved by the energy method which involves the derivation of the expressions for the kinetic and potential energies. The effect of rotation on flexural vibrations is so significant that it cannot be neglected.

Denote by r and θ the polar co-ordinates of an arbitrary point in the middle plane and by $w = w(r, \theta)$ the deflection at that point. If, as before, ρ is the density of the material and $h = h(r)$ the thickness of the disk, then the mass of the element is $\rho hr\,dr d\theta$ and its kinetic energy is $\frac{\dot{w}^2}{2}\rho hr dr\,d\theta$. The total *kinetic energy* is

$$T = \frac{\rho}{2}\int_0^{2\pi}\int_{r_1}^{r_2}[\dot{w}(r, \theta)]^2 h(r)\,r\,dr\,d\theta. \tag{11.1}$$

For *umbrella vibrations*, the deflection w is independent of the co-ordinate θ and, in place of expression (11.1), we obtain

$$T = \pi\rho\int_{r_1}^{r_2}[\dot{w}(r)]^2 h(r)\,r\,dr. \tag{11.2}$$

If there are distributed masses (blades) at the contour of the disk, their kinetic energy must also be taken into account.

We present, without derivation, the expression for the *potential energy* of a circular plate due to bending

$$U = \frac{E}{24(1-\mu^2)}\int_0^{2\pi}\int_{r_1}^{r_2}\left\{\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 + \right.$$

$$\left. + 2(1-\mu)\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right]^2 - 2(1-\mu)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right\} hr\,dr\,d\theta.$$

In the case of *umbrella vibrations* the deflection w is independent of the co-ordinate θ and the last expression is written as

$$U = \frac{\pi E}{12(1-\mu^2)}\int_{r_1}^{r_2}\left[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)^2 - \frac{2(1-\mu)}{r}\frac{\partial^2 w}{\partial r^2}\frac{\partial w}{\partial r}\right] hr\,dr. \tag{11.3}$$

Because of bending, each element of the disk somewhat approaches the axis of rotation and in consequence accumulates some additional potential energy as, due to rotation of the disk, the bending occurs in the centrifugal field. This phenomenon was considered above in relation to flexural vibrations of a stretched bar (stretched by a centrifugal force, for example). In the present case, to an elementary mass $\rho hr d\theta\, dr$ corresponds a centrifugal force $\omega^2 \rho h r^2\, d\theta\, dr$. The additional energy is $u\omega^2 \rho h r^2 d\theta dr$, where u is the radial displacement of the element which is expressed in terms of the deflection as

$$u = \frac{1}{2} \int_{r_1}^{r_2} \left(\frac{\partial w}{\partial r}\right)^2 dr.$$

Thus, the total potential energy for the centrifugal field is

$$U_* = \frac{1}{2} \rho\omega^2 \int_0^{2\pi} \int_{r_1}^{r_2} \left[\int_{r_1}^{r} \left(\frac{\partial w}{\partial r}\right)^2 dr \right] r^2 h\, d\theta\, dr.$$

In the case of umbrella vibrations in which w is independent of θ this formula becomes

$$U_* = \pi\rho\omega^2 \int_{r_1}^{r_2} \left[\int_{r_1}^{r} \left(\frac{\partial w}{\partial r}\right)^2 dr \right] r^2 h\, dr. \tag{11.4}$$

Umbrella Vibrations

In the case of free umbrella vibrations the deflection w is independent of the polar angle θ and can be represented as

$$w = R(r) \sin pt. \tag{11.5}$$

According to formulas (11.2) and (11.5) the maximum kinetic energy is

$$T_{\max} = \pi\rho p^2 \int_{r_1}^{r_2} R^2 hr\, dr. \tag{11.6}$$

The maximum potential energy of strain according to expressions (11.3) and (11.5) is

$$U_{\max} = \frac{\pi E}{12(1-\mu^2)} \int_{r_1}^{r_2} \left[\left(R'' + \frac{R'}{r}\right)^2 - \frac{2(1-\mu)}{r} R'R'' \right] hr\, dr. \tag{11.7}$$

Using expression (11.4), we find

$$U_{*\max} = \pi\rho\omega^2 \int_{r_1}^{r_2} \left[\int_{r_1}^{r} (R')^2\, dr \right] r^2 h\, dr. \tag{11.8}$$

Based on the law of conservation of energy, we can write *

$$T_{\max} = U_{\max} + U_{\omega\max};$$

substituting expressions (11.6) to (11.8) and solving the equation for the square of the frequency, we find

$$p^2 = \left\{ \frac{E}{12\rho(1-\mu^2)} \int_{r_1}^{r_2} \left[\left(R'' + \frac{R'}{r} \right)^2 - \frac{2(1-\mu)}{r} R'R'' \right] hr\,dr + \right.$$

$$\left. + \omega^2 \int_{r_1}^{r_2} \left[\int_{r_1}^{r} (R')^2\,dr \right] r^2 h\,dr \right\} : \left\{ \int_{r_1}^{r_2} R^2 hr\,dr \right\}. \qquad (11.9)$$

The substitution of a suitable function $R(r)$ will enable one to calculate without great difficulty the approximate value of the lowest frequency. This function must satisfy at least the geometric boundary conditions at $r = r_1$

$$R = 0, \ R' = 0. \qquad (11.10)$$

Compare the results of using four different choices of the function $R(r)$ for a disk of uniform thickness. All the variants satisfy conditions (11.10) and correspond to the simplest mode of vibration involving no nodal circles.

First variant. $R = (r - r_1)^2$ gives a value for p^2 that is 49 per cent too high, as compared with the exact value.

Second variant. $R = (r - r_1)^s$ contains one undetermined parameter; its value is chosen so as to obtain the minimum value of p^2. It is found that $s = 1.35$ and the error in the calculated square of the frequency is 7.6 per cent.

Third variant

$$R = (r - r_1)^2 [1 + a(r - r_1)]$$

also contains an undetermined parameter a. The minimum condition for the frequency gives a value $a = -0.463$ and leads to an error for p^2 even smaller than in the second variant (6 per cent).

The foregoing variants have one disadvantage in common (though to a different degree): while corresponding to the kinematic conditions at the inner contour they do not satisfy the *force conditions* at the outer contour.

A radical improvement is attained by using the fourth variant

$$R = (r - r_1)^2 [1 + a(r - r_1) + b(r - r_1)^2],$$

---

* In a more accurate solution it is necessary also to include the work done during flexural vibrations by hoop forces (acting in the middle surface of the disk).

where the coefficients a and b are chosen so as to satisfy the conditions at the outer contour as well. The error in the result is then only 1.6 per cent.

This example demonstrates the importance of satisfying as many boundary conditions as possible when using not only the Bubnov-Galerkin method but also Rayleigh's method.

Fan Vibrations

In this case, in place of expression (11.5) we have to assume for the deflection w a function which reflects also the dependence of the deflection on the polar angle θ. Bearing in mind the periodicity of this dependence, we can assume

$$w = R(r) \cos n\theta \sin pt,$$

where $n = 1, 2, \ldots$ is the number of *nodal diameters* whose points are not vibrating; to each mode of vibration corresponds a definite value of the frequency.

Further solution is carried out in the same manner as above. As a result, we obtain a formula similar to (11.9) but containing also the number n of nodal diameters. In this case the application of different variants of the function $R(r)$ leads to very close results which are in good agreement with the exact values (the error is about 1 per cent).

This difference from the case of umbrella vibrations is no mere chance. The point is that the approximate mode for umbrella vibrations *satisfies* the conditions only in the root and peripheral regions whereas for fan vibrations it satisfies the conditions along all the nodal diameters as well.

12. Flexural Vibrations of Rectangular Plates

In the general case of a plate of variable thickness the solution in closed form is not possible; for such problems use can be made of the energy method. If a plate has a uniform thickness, the problem becomes less complicated and admits a relatively simple solution for some boundary conditions.

Plate of Uniform Thickness (Exact Solution)

If a plate carries a static distributed load q and the deflections w of points of the middle surface are small, the following differential equation applies

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}, \tag{12.1}$$

where the *flexural rigidity* of the plate is

$$D=\frac{Eh^3}{12(1-\mu^2)}\,.$$

In problems involving free vibrations, only the inertia forces constitute the load

$$q=-\rho h\frac{\partial^2 w}{\partial t^2}\,, \tag{12.2}$$

where ρ = density of material,
h = thickness of plate.

Substituting expression (12.2) in Eq. (12.1), we arrive at the basic differential equation

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}=-\rho h\frac{\partial^2 w}{\partial t^2}\,.$$

Representing, as in the preceding cases, the solution in the form

$$w=W(x, y)\,T(t),$$

we obtain the differential equation for the mode of vibration $W(x, y)$

$$\frac{\partial^4 W}{\partial x^4}+2\frac{\partial^4 W}{\partial x^2\,\partial y^2}+\frac{\partial^4 W}{\partial y^4}=\frac{\rho h p^2}{D}W; \tag{12.3}$$

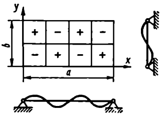

Fig. 73

in the case of a rectangular plate with simply supported edges this differential equation has the particular solution

$$W=A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\qquad(m=1, 2, \ldots;\ n=1, 2, \ldots), \tag{12.4}$$

where a and b are the sides of the plate.

Relation (12.4) is illustrated in Fig. 73. During vibration any straight line parallel to the x axis deforms into a sine curve involving m half-waves in the interval $(0, a)$($m = 4$ in Fig. 73); likewise, straight lines parallel to the y axis become sine curves with n half-waves ($n = 2$ in Fig. 73).

Expression (12.4) satisfies the boundary conditions at the contour (zero deflections and zero bending moments)

$$W=0,\qquad \frac{\partial^2 W}{\partial x^2}+\mu\frac{\partial^2 W}{\partial y^2}=0 \text{ at } x=0 \text{ and } x=a,$$

$$W=0,\qquad \frac{\partial^2 W}{\partial y^3}+\mu\frac{\partial^2 W}{\partial x^2}=0 \text{ at } y=0 \text{ and } y=b.$$

Substituting expression (12.4) in Eq. (12.3), we obtain

$$\left(\frac{m\pi}{a}\right)^4+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4=\frac{\rho h p^2}{D}\,;$$

hence we find the natural frequency

$$p_{mn}=\frac{\pi^2 h}{2}\sqrt{\frac{E}{3\rho(1-\mu^2)}}\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right].$$

The frequency depends on the numbers m and n defining the number of half-waves into which the plate is subdivided in either direction. The lowest frequency corresponds to the case when the plate is bent in one half-wave in either direction ($m = 1$, $n = 1$)

$$p_{11}=\frac{\pi^2 h}{2}\sqrt{\frac{E}{3\rho(1-\mu^2)}}\left(\frac{1}{a^2}+\frac{1}{b^2}\right). \tag{12.5}$$

If one of the sides of the plate is considerably larger than the other, one term in the parentheses becomes very small as compared to the other and vanishes in the limit. Let, for instance, $a/b \to \infty$; formula (12.5) then takes the form

$$p_{11}=\frac{\pi^2 h}{2b^2}\sqrt{\frac{E}{3\rho(1-\mu^2)}}. \tag{12.6}$$

In this case the middle surface of the plate is bent to cylindrical form during vibration; it may be said that the plate is made up of a number of identical (and identically bent) beam-strips of span b. If we assume that these beam-strips do not interact with each other, their natural frequency can be found by formula (8.32a) substituting the moment of inertia of the cross section $I = h^3/12$ (the width of the beam-strip may be taken equal to any number, say, unity) and the intensity of the distributed mass $m = \rho h$. The natural frequency is then given by expression (12.6) but with no divider $1 - \mu^2$ under the radical. This difference is due to the fact that the lateral deformation of a beam-strip forming part of the plate is restrained by adjacent beam-strips whereas an isolated beam-strip is free of all restraints.

The foregoing solution owes its simplicity not only to the simple shape of the plate but also to the boundary conditions. For other modes of fixing the edges, we have to resort to approximate methods for solving the differential equation (12.3).

Approximate Solution

The application of Rayleigh's or Ritz's method involves the use of the expression for the potential energy of a rectangular plate due to bending at the time of maximum deflection

$$U_{\max}=\frac{1}{2}\int_0^a\int_0^b D\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2+\left(\frac{\partial^2 W}{\partial y^2}\right)^2+2\mu\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+\right.$$
$$\left.+2(1-\mu)\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]dx\,dy. \tag{12.7}$$

The maximum kinetic energy of an element of the plate $\frac{\rho h\,(\dot{w})^2_{\max}}{2}\,dx\,dy$ upon substitution $w = W \sin pt$ assumes the form $\frac{\rho h p^2}{2} W^2 dx\,dy$. The total kinetic energy of the whole plate is

$$T_{\max} = \frac{\rho p^2}{2} \int\limits_0^a \int\limits_0^b W^2 h\,dx\,dy. \tag{12.8}$$

Equating expressions (12.7) and (12.8), we obtain

$$p^2 = \left\{ \int\limits_0^a \int\limits_0^b D \left[\left(\frac{\partial^2 W}{\partial x^2} \right) + \left(\frac{\partial^2 W}{\partial y^2} \right)^2 + 2\mu \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} + \right. \right.$$
$$\left. \left. + 2(1-\mu) \left(\frac{\partial^2 W}{\partial x\,\partial y} \right)^2 \right] dx\,dy \right\} : \rho \left[\int\limits_0^a \int\limits_0^b W^2 h\,dx\,dy \right]. \tag{12.9}$$

By suitably assigning the form of the function $W(x, y)$ an approximate value of the square of the natural frequency can be found from formula (12.9).

CHAPTER III

CRITICAL STATES OF ROTATING SHAFTS AND ROTORS

13. Shaft with One Disk

The rotation of an unbalanced shaft is always accompanied by more or less intensive transverse vibrations. The amplitudes of vibration depend on the speed of rotation and at certain critical values of the speed characteristic of a given shaft they build up so violently that upset the normal service conditions and may cause rupture of the shaft. The critical state cannot be avoided even by very thorough balancing. Care must therefore be taken to keep the operating speeds far from critical values.

Critical Speed

Consider a shaft with a disk of mass m mounted on it with eccentricity e. In order to eliminate the effect of gravity and consider the phenomenon in the purest form, we assume that the axis of the shaft is vertical (Fig. 74). The shaft has a circular section and rotates in bearings; the disk is located halfway between the supports.*

When the shaft rotates with an angular velocity ω, the centre of gravity of the disk moves in a circle and a centrifugal force develops. If the deflection of the shaft produced by this force is denoted by r, the resultant eccentricity is found to be $e + r$ and, consequently, the centrifugal force is $m\omega^2 (e + r)$. In order to determine the deflection r, the centrifugal force should be divided by the bending stiffness of the shaft k

$$r = \frac{m\omega^2 (e+r)}{k};$$

hence we find that the deflection of the shaft is proportional to the initial eccentricity

$$r = \frac{m\omega^2 e}{k - m\omega^2}. \qquad (13.1)$$

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* The subsequent arguments hold true also for a more general arrangement of supports provided the stiffness of the shaft is the same in all directions and the plane of the disk remains perpendicular to the axis of rotation when the shaft is bent.

An important conclusion may be drawn from formula (13.1): if the equality $k = m\omega^2$ is fulfilled, the denominator on the right-hand side of formula (13.1) is zero and the deflection becomes infinite. The critical state occurs at a definite value of the angular velocity

$$\omega_{cr} = \sqrt{\frac{k}{m}}, \tag{13.2}$$

which, as can be seen, depends on the parameters of the system. This speed is called a *critical speed of rotation*; it coincides with the

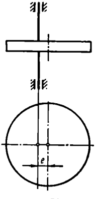


Fig. 74

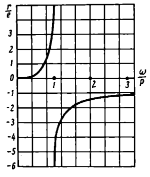


Fig. 75

natural frequency p of transverse vibration of a non-rotating shaft-disk system; the stiffer the shaft and the lighter the disk, the higher is the critical speed.

From formula (13.2) we can obtain an expression for the relative deflection of the shaft

$$\frac{r}{e} = \frac{1}{\omega_{cr}^2/\omega^2 - 1}. \tag{13.3}$$

The curve obtained by plotting r/e against the ratio $\omega/\omega_{cr} = \omega/p$ is shown in Fig. 75. The analysis of formula (13.3) and Fig. 75 indicates that when the rotation is slow the deflections r are small and increase with increasing speed; the centre of gravity S of the disk is farther from the centre of rotation O than the centre W of the section of the shaft (Fig. 76*a*). If $\omega/\omega_{cr} = 1$, the deflection becomes infinite (critical state).

In the postcritical region when $\omega > \omega_{cr}$ the deflections are again finite but opposite in sign to the initial eccentricity. Figure 76*b* shows the relative positions of the centres S, O and W corresponding to this case. When the rotation is fast ($\omega > \omega_{cr}$), the centre of gravity S of the disk is closer to the centre of rotation O than the centre W of the shaft. The higher the speed, the closer is the centre of gravity S of the disk to the centre of rotation O; as $\omega \to \infty$ the centre of gravity of the disk approaches indefinitely the axis of rotation. Thus, at very high speeds *the disk is self-centred*. Therefore, making the shaft very flexible (i.e., providing small values of ω_{cr}), it is possible to obtain a well balanced system; this is used in designing shafts of high-speed turbines where flexible shafts are found to be more rational than rigid ones.

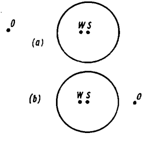


Fig. 76

The critical state was defined above as the state in which the deflections of the shaft increase without limit if the disk has an initial eccentricity. Another possible interpretation of the critical state will be used in what follows. From expression (13.1) it is seen that if $e = 0$ and at the same time $k = m\omega^2$, the deflection r is *indeterminate*. This means that at $\omega = \omega_{cr}$ a completely balanced shaft *loses the stability* of straight-line form. If this form is disturbed (for example, by impact), the shaft has no tendency to restore it; the point is that the elastic reaction kr is exactly balanced by the centrifugal force $m\omega_{cr}^2 r$ due to the deflection r.

At any fixed value of the speed (except $\omega = \omega_{cr}$) the rotation is accompanied by a definite and time-independent deformation of the shaft. During the motion any fibre remains stretched (or compressed) the same amount regardless of time. In this sense the phenomenon under consideration can hardly be regarded as an elastic vibration though for a fixed observer the motion looks like a vibration in two directions.

The critical state is usually considered impermissible in service and it is customary to indicate a forbidden range of dangerous values of speeds near ω_{cr} (for example, between 0.7 ω_{cr} and 1.4 ω_{cr}).

Free Vibrations about Steady-State Position

As previously noted, in the steady-state motion ($\omega \neq \omega_{cr}$) there are no elastic vibrations in the usual sense as the stresses and strains at any point in each section of the shaft are unchanged. Of some interest is the process of free vibration *about* the steady-state position.

To investigate this process we revert to the simplest scheme (see Fig. 74) but, for simplicity, we assume $e = 0$. Then, at $\omega \neq \omega_{cr}$ to the steady state corresponds the rotation of a straight, undeformed shaft. Suppose that the shaft is bent due to some disturbance and determine the subsequent motion in a moving system of co-ordinates.

Figure 77*a* shows the position of the centre of rotation O and the instantaneous position of the centre of gravity S of the disk. The

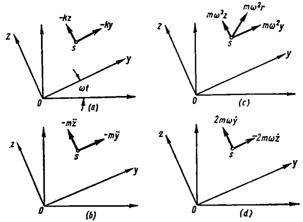


Fig. 77

co-ordinate system yz rotates with a uniform angular velocity ω equal to the speed of the disk; the latter will be considered constant (this can always be provided by proper variation of the torque).

During the motion the disk is acted on by an elastic reaction of the shaft equal to kr; Figure 77*a* shows its projections on the y and z axes equal to $-ky$ and $-kz$, respectively. The projections of the inertia forces of the disk are represented in separate diagrams (Fig. 77*b* to *d*): the projections of the inertia force in relative motion, $-m\ddot{y}$ and $-m\ddot{z}$; the projections of the inertia force in base motion, $m\omega^2 y$ and $m\omega^2 z$; the projections of the Coriolis force, $-2m\omega\dot{z}$ and $2m\omega\dot{y}$.

The equations of motion of the disk are

$$\left.\begin{aligned} -ky - m\ddot{y} + m\omega^2 y - 2m\omega\dot{z} &= 0, \\ -kz - m\ddot{z} + m\omega^2 z + 2m\omega\dot{y} &= 0. \end{aligned}\right\} \tag{13.4}$$

Using the relation $k = mp^2$ (where p is the natural frequency of the non-rotating system, $p = \omega_{cr}$), we obtain the basic system of equations

$$\left.\begin{aligned} \ddot{y} + (p^2 - \omega^2)\, y + 2\omega\dot{z} &= 0, \\ \ddot{z} + (p^2 - \omega^2)\, z - 2\omega\dot{y} &= 0. \end{aligned}\right\} \tag{13.5}$$

When $\omega \neq p$ its particular solution can be assumed in the form

$$y = a_1 \sin(\lambda t + \alpha), \quad z = a_2 \cos(\lambda t + \alpha). \tag{13.6}$$

Substituting expressions (13.6) in Eqs. (13.5), we obtain

$$\left.\begin{aligned} -a_1\lambda^2 + (p^2 - \omega^2)\, a_1 - 2\omega a_2 \lambda &= 0, \\ -a_2\lambda^2 + (p^2 - \omega^2)\, a_2 - 2\omega a_1 \lambda &= 0. \end{aligned}\right\} \tag{13.7}$$

This homogeneous system has non-vanishing roots only if the determinant of the coefficients of the system is zero

$$\begin{vmatrix} p^2 - \omega^2 - \lambda^2 & -2\omega\lambda \\ -2\omega\lambda & p^2 - \omega^2 - \lambda^2 \end{vmatrix} = 0,$$

i.e.,

$$[(p^2 - \omega^2) - \lambda^2]^2 - (2\omega\lambda)^2 = 0.$$

Hence we find two roots for λ^2

$$\lambda_1^2 = (p + \omega)^2, \quad \lambda_2^2 = (p - \omega)^2.$$

Substituting each of these roots in either of Eqs. (13.7) successively, we can find the relation between the amplitudes a_1 and a_2. To the first root corresponds the equality $a_1 = a_2$ and to the second, $a_1 = -a_2$.

The general solution of system (13.5) must therefore be written as

$$\begin{aligned} y &= A_1 \sin[(p + \omega)\, t + \alpha_1] + A_2 \sin[(p - \omega)\, t + \alpha_2], \\ z &= A_1 \cos[(p + \omega)\, t + \alpha_1] - A_2 \cos[(p - \omega)\, t + \alpha_2]. \end{aligned}$$

The constants A_1, A_2, α_1 and α_2 are determined by the initial conditions of the motion.

Thus, any disturbance leads to elastic vibrations with constant amplitude; this is indicative of the stability of the undisturbed state of equilibrium both at $\omega < p$ and at $\omega > p$ (the case of $\omega = p$

is excluded from consideration for the present). It might be well to note that the frequencies of both vibrations, $\omega + p$ and $\omega - p$, observed in the rotating system of co-ordinates, differ from the natural frequency of vibration of the non-rotating system.

In the particular case when $\omega = p$ one of the frequencies vanishes and the corresponding component of the motion is no longer a vibratory one; this points to the *instability* of the undisturbed state of equilibrium.

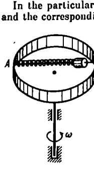


Fig. 78

The system represented in Fig. 78 possesses somewhat different properties. In this system, which also rotates with an angular velocity ω, an elastically restrained mass m may slide along a guide AB; in a rotating system of coordinates, $z = 0$ and the motion is defined by a single function $y = y(t)$. Assume that the stiffness of the spring is k and the position of the mass on the axis of rotation corresponds to the state of equilibrium. We investigate the properties of the motion of the mass when this state is upset.

Assume that at a current instant the mass is at a distance y from the axis of rotation. In writing the differential equation of relative motion of the mass it is necessary to take into account the spring force $-ky$ and the inertia force in base motion $m\omega^2 y$; we thus obtain

$$\left.\begin{aligned} -ky + m\omega^2 y &= m\ddot{y}, \\ \text{i.e.,} \qquad \ddot{y} + \left(\frac{k}{m} - \omega^2\right) y &= 0. \end{aligned}\right\} \qquad (13.8)$$

From this way of writing the equation it is immediately seen that at $\omega^2 < k/m$ the motion will be a vibration

$$y = a \sin\left(\sqrt{\frac{k}{m} - \omega^2}\, t + \alpha\right)$$

with a frequency $\sqrt{k/m - \omega^2}$ lower than the frequency of vibration of the mass $p = \sqrt{k/m}$ without rotation.

Insofar as the disturbed motion is a vibration with constant amplitude, the undisturbed state of equilibrium of the mass (on the axis of rotation) should be considered *stable*.

Let us now turn to the case when

$$\omega^2 > \frac{k}{m}.$$

The solution of the differential equation (13.8) assumes then the form

$$y = a \sinh\left(\sqrt{\omega^2 - \frac{k}{m}}\, t + \alpha\right),$$

i.e., we have an *aperiodic motion* of the system *away* from the origin of co-ordinates. This means that the undisturbed state of equilibrium is *unstable* in the present case.

We may finally conclude that the system under consideration is stable only at angular velocities lower than the natural frequency of vibration of the non-rotating system. It will be recalled that a disk mounted on a shaft (see Fig. 74) is stable at any values of the angular velocity ω different from ω_{cr}, both at $\omega < \omega_{cr}$ and at $\omega > \omega_{cr}$; this is the basic difference between the systems of Figs. 74 and 78.

The stability of the system in Fig. 74 at $\omega > \omega_{cr}$ is due to the relative motion in the z direction and the associated Coriolis forces acting along the y axis.

Gyroscopic Effect

We have considered the case when a rotating disk remains in the same plane at all times. This is true, for instance, of a two-support shaft with a disk placed halfway between the supports and is a direct consequence of symmetry. In most cases, however, the plane of the disk changes its orientation during the motion; this is accompanied by specific phenomena which are known as *gyroscopic effects*.

Let us first consider an auxiliary problem of a rigid body consisting of a thin disk and a shaft and rotating about an axis AA_1 which makes a small angle α with the shaft axis (Fig. 79*a*): the angular velocity is ω and the distance Δ from the centre O of the disk to the axis of rotation is constant.

In this motion any point of the disk describes a circular path whose radius r is equal to the distance from the point to the axis AA_1; accordingly an element of mass dm develops a centrifugal force $\omega^2 r\,dm$.

Attach the axes of a moving co-ordinate system xy to the centre O of the disk. The centrifugal force $\omega^2 r\,dm$ is parallel to the xy plane and can be resolved into components $\omega^2 (x + \Delta)\, dm$ (parallel to the x axis) and $\omega^2 y\,dm$ (parallel to the y axis). Reducing the set of all centrifugal forces to centre O, we obtain the components of the resultant inertia force

$$P_x = \int\limits_{(m)} \omega^2 (x + \Delta)\, dm = m\omega^2\Delta,$$

$$P_y = \int\limits_{(m)} \omega^2 y\, dm = 0,$$

i.e., the resultant inertia force is directed along the x axis and has no components along the other axes.

Consider the determination of the moments of inertia forces with respect to the x and y axes. The point of application of an elementary centrifugal force is at a distance αx from the xy plane.

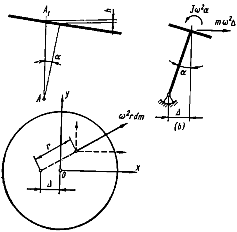


Fig. 79

Hence the moments of these forces with respect to the above axes are

$$M_x = \int_{(m)} x\alpha\omega^2 y\,dm = 0,$$

$$M_y = -\int_{(m)} x\alpha\omega^2 (x+\Delta)\,dm = -J_y\omega^2\alpha, \tag{13.9}$$

where J_y is the moment of inertia of the disk about its diameter.

Figure 79b shows the action of the force P_x and the moment M_y. It is important to note that the moment M_y tends to bring the shaft axis into coincidence with the axis of rotation; this is an essential feature of the gyroscopic effect.

We can now direct our attention to the basic problem. Usually the distortion of the disk *is not assigned* but results from the deflection of the shaft, as when the disk is located off the mid-span (Fig. 80a). Assume no initial eccentricity and determine the critical speed from the condition of equilibrium of the deflected axis of the shaft. Let r be the deflection of the shaft axis at the section where the disk is attached; α the angle of rotation of this section. The shaft is acted on by a centrifugal force $m\omega_{cr}^2 r$ and a moment $-J_y\omega_{cr}^2\alpha$. We make use of unit displacements (Fig. 80*b* and *c*):

δ_{rr} = deflection of shaft at the plane of disk due to unit centrifugal force,

$\delta_{r\alpha}$ = deflection of shaft at the plane of disk due to unit moment,

$\delta_{\alpha r}$ = angle of rotation of the plane of disk due to unit centrifugal force ($\delta_{\alpha r} = \delta_{r\alpha}$),

$\delta_{\alpha\alpha}$ = angle of rotation of the plane of disk due to unit moment.

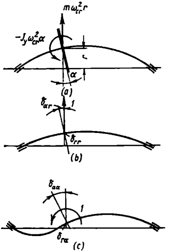


Fig. 80

Using the unit displacements, we can write the total displacements r and α as

$$r = m\omega_{cr}^2 r\delta_{rr} - J_y\omega_{cr}^2\alpha\delta_{r\alpha},$$

$$\alpha = m\omega_{cr}^2 r\delta_{\alpha r} - J_y\omega_{cr}^2\alpha\delta_{\alpha\alpha}$$

or

$$\left.\begin{aligned} r(1 - m\omega_{cr}^2\delta_{rr}) + J_y\omega_{cr}^2\delta_{r\alpha}\alpha &= 0, \\ -rm\omega_{cr}^2\delta_{\alpha r} + \alpha(1 + J_y\omega_{cr}^2\delta_{\alpha\alpha}) &= 0. \end{aligned}\right\} \qquad (13.10)$$

The two algebraic equations obtained are homogeneous in the unknowns (deflection r and angle α); hence, in order to have displacements r and α different from zero, the determinant of the coefficients of the system of equations (13.10) must be zero

$$\begin{vmatrix} 1 - m\omega_{cr}^2\delta_{rr} & J_y\omega_{cr}^2\delta_{r\alpha} \\ -m\omega_{cr}^2\delta_{\alpha r} & 1 + J_y\omega_{cr}^2\delta_{\alpha\alpha} \end{vmatrix} = 0. \qquad (13.11)$$

Expanding the determinant, we obtain a biquadratic equation for the critical speed

$$\omega_{cr}^4 + \frac{m\delta_{rr} - J_y\delta_{\alpha\alpha}}{mJ_y(\delta_{\alpha\alpha}\delta_{rr} - \delta_{\alpha r}^2)}\omega_{cr}^2 - \frac{1}{mJ_y(\delta_{\alpha\alpha}\delta_{rr} - \delta_{\alpha r}^2)} = 0. \qquad (13.12)$$

Equation (13.10) has only one positive root for ω_{cr}^2 and therefore determines *a single value of the critical speed*. If the actual speed differs from ω_{cr}, the determinant of the coefficients of system (13.11) will not be zero, and hence $r = 0$ and $\alpha = 0$.

Let us consider in detail the case of an overhanging shaft. In this case

$$\delta_{rr} = \frac{l^3}{3EI}, \qquad \delta_{r\alpha} = \delta_{\alpha r} = \frac{l^2}{2EI}, \quad \delta_{\alpha\alpha} = \frac{l}{EI}$$

and Eq. (13.12) becomes

$$\omega_{cr}^4 + \frac{12EI}{mJ_y l^3}\left(\frac{ml^2}{3} - J_y\right)\omega_{cr}^2 - \frac{12E^2I^2}{mJ_y l^4} = 0. \qquad (13.13)$$

For further analysis it is convenient to introduce the dimensionless parameters

$$K = \frac{m\omega_{cr}^2 l^3}{EI}, \quad D = \frac{J_y}{ml^2}.$$

In place of Eq. (13.13) we then obtain

$$K^2 + 4K\left(\frac{1}{D} - 3\right) - \frac{12}{D} = 0,$$

i.e.,

$$K = 2\left[\left(3 - \frac{1}{D}\right) + \sqrt{\left(3 - \frac{1}{D}\right)^2 + \frac{3}{D}}\right]$$

(the negative sign before the radical is left out as leading to a physically impossible result). Figure 81 shows the graph $K = K(D)$.

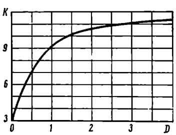


Fig. 81

The limiting cases are $D = 0$ (inertialess disk) and $D \to \infty$ (disk of infinite rotary inertia); in the former case we have the common result ($\omega_{cr}^2 = 3EI/ml^3$) and in the latter, the result corresponding to a disk which maintains the same plane of rotation.

Example 17. A two-support shaft (Fig. 82) with overhang carries a thin disk of diameter $0.6a$ at its free end (a is the span of the shaft); the length of the overhang is $0.5a$. Determine the critical speed of the shaft when the gyroscopic effect is taken into account.

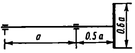


Fig. 82

We first find the unit displacements

$$\delta_{rr}=0.125\frac{a^3}{EI}, \quad \delta_{\alpha r}=\delta_{r\alpha}=0.292\frac{a^2}{EI},$$

$$\delta_{\alpha\alpha}=0.833\frac{a}{EI}.$$

Further we calculate the mass moment of inertia of the disk

$$J_y=\frac{m\,(0.6a)^2}{16}=0.0225ma^2.$$

Equation (13.12) becomes

$$\omega_{cr}^4+250\frac{EI}{ma^3}\,\omega_{cr}^2-2{,}350\left(\frac{EI}{ma^3}\right)^2=0.$$

hence

$$\omega_{cr}^2=9.1\frac{EI}{ma^3}, \quad \omega_{cr}=\frac{3.02}{a}\sqrt{\frac{EI}{ma}}.$$

If the gyroscopic effect is neglected, we obtain from the elementary formula

$$\omega_{cr}=p=\sqrt{\frac{1}{m\delta_{rr}}}=\frac{2.83}{a}\sqrt{\frac{EI}{ma}}.$$

Thus, the gyroscopic effect increases the critical speed by nearly 7 per cent. The increased critical speed is the result of the larger stiffness of the system due to the resistance offered by the disk to a change of the plane of rotation.

Critical Speeds of a Shaft Whose Section Has Different Principal Moments of Inertia

Consider the case when the section of a shaft has different principal moments of inertia (shaft with keyways or flats, rotor shaft of a double-pole electric machine with longitudinal slots for the winding, etc.). Assume no initial eccentricity and neglect the gyroscopic effect; these simplifications help in revealing the effect of the main feature—the difference in the flexural rigidities of the shaft. The angular velocities of the shaft and disk are assumed to be the same and constant in time; as pointed out above, this can be provided by proper variation of the external torque.

We follow the same line of reasoning as above and consider the disturbed position of the section shown in Fig. 83. Here, a rotating co-ordinate system should be chosen more specifically. In discussing the diagram of Fig. 77, the coordinate system xy was assumed to rotate with the same angular velocity ω as the disk; we now further assume that the y and z axes are chosen parallel to the principal axes of inertia of the shaft section (*1-1* and *2-2*). The parallelism will be maintained at all times as the angular velocities of the disk and the co-ordinate system are the same. In this case the equations of motion of the disk will be of the form of Eqs. (13.4), but instead of the stiffness k we have to deal with two stiffnesses, k_1 and k_2, which are different for axes *1-1* and *2-2*. We obtain then the system

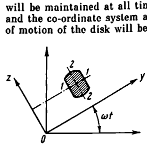


Fig. 83

$$\left.\begin{aligned} -k_1y-m\ddot{y}+m\omega^2y-2m\omega\dot{z}&=0,\\ -k_2z-m\ddot{z}+m\omega^2z+2m\omega\dot{y}&=0.\end{aligned}\right\} \tag{13.14}$$

Replace k_1 and k_2 using the formulas

$$k_1=mp_1^2,\quad k_2=mp_2^2, \tag{13.15}$$

where p_1 and p_2 are the natural frequencies of vibration of the non-rotating system.

Similarly to Eqs. (13.5) we then obtain

$$\left.\begin{aligned} \ddot{y}+(p_1^2-\omega^2)\,y+2\omega\dot{z}&=0,\\ \ddot{z}+(p_2^2-\omega^2)\,z-2\omega\dot{y}&=0.\end{aligned}\right\} \tag{13.16}$$

We are primarily concerned with determining the angular velocities ω at which the critical state occurs. An inspection of system (13.16) indicates that the critical states correspond to the angular velocities $\omega=p_1$ and $\omega=p_2$. In the first case we have the solution $y=$ constant and $z=0$, and in the second case, $y=0$ and $z=$ constant. Both solutions define the critical state since the deflection of the shaft is maintained with no initial eccentricity e.

It remains to be seen what the state of the shaft is at angular velocities ω lying in the interval $[p_1, p_2]$. We shall prove that the critical states correspond to this entire interval.

The solution to the system of equations (13.16) is sought in the form

$$y=a_1e^{\lambda t},\quad z=a_2e^{\lambda t}. \tag{13.17}$$

Substituting expressions (13.17) in Eqs. (13.16), we obtain two homogeneous equations in a_1 and a_2

$$\begin{aligned} (\lambda^2+p_1^2-\omega^2)\,a_1+2\omega\lambda a_2&=0,\\ -2\omega\lambda a_1+(\lambda^2+p_2^2-\omega^2)\,a_2&=0.\end{aligned}$$

The condition of non-zero solutions for a_1 and a_2 is

$$\begin{vmatrix} \lambda^2 + p_1^2 - \omega^2 & +2\omega\lambda \\ -2\omega\lambda & \lambda^2 + p_2^2 - \omega^2 \end{vmatrix} = 0,$$

i.e.,

$$\lambda^4 + \lambda^2 (p_1^2 + p_2^2 + 2\omega^2) - (p_1^2 - \omega^2)(p_2^2 - \omega^2) = 0.$$

Solving this biquadratic equation, we obtain two real roots for λ^2

$$\lambda^2 = \frac{1}{2}\left[-(p_1^2 + p_2^2 + 2\omega^2) \pm \sqrt{(p_1^2 - p_2^2)^2 + 8\omega^2 (p_1^2 + p_2^2)}\right].$$

Therefore all roots λ_i ($i = 1, 2, 3, 4$) are pure imaginary or real. To imaginary roots corresponds a vibratory motion with constant amplitudes, and to negative real roots corresponds an aperiodic damped process. If, however, there is a positive root λ, the displacements y and z will approach infinity in the course of time, i.e., the initial state is *unstable*.

But in order to have one of the roots positive, the following inequality must be fulfilled

$$\sqrt{(p_1^2 - p_2^2)^2 + 8\omega^2 (p_1^2 + p_2^2)} > p_1^2 + p_2^2 + 2\omega^2. \tag{13.18}$$

Squaring both sides of inequality (13.18) and cancelling like terms, we obtain

$$(\omega^2 - p_1^2)(\omega^2 - p_2^2) < 0.$$

This inequality is satisfied if ω lies within the limits $p_1 \leqslant \omega \leqslant p_2$, which proves the instability of the shaft over this range of angular velocities.

From the above discussion it follows that the difference in the stiffnesses k_1 and k_2 *increases the risk* of running into critical states.

Effect of Gravity on a Horizontal Shaft

If the x axis of the shaft in Fig. 83 is horizontal, the effect of gravity must be taken into account in setting up the equations of motion; the projections of the force of gravity are, respectively, $-mg \sin \omega t$ and $-mg \cos \omega t$. Instead of Eqs. (13.14) we then obtain

$$-k_1 y - m\ddot{y} + m\omega^2 y - 2m\omega\dot{z} - mg \sin \omega t = 0,$$

$$-k_2 z - m\ddot{z} + m\omega^2 z + 2m\omega\dot{y} - mg \cos \omega t = 0.$$

Replacing k_1 and k_2 from formulas (13.15), we arrive at a *non-homogeneous system* of equations

$$\left.\begin{aligned} \ddot{y} + (p_1^2 - \omega^2)\, y + 2\omega\dot{z} &= -g \sin \omega t, \\ \ddot{z} + (p_2^2 - \omega^2)\, z - 2\omega\dot{y} &= -g \cos \omega t \end{aligned}\right\} \tag{13.19}$$

which differs from system (13.16) by non-zero right-hand sides.

The general solution of system (13.19) is made up of the sum of the solution of the homogeneous problem (13.16) and a particular solution of Eqs. (13.19). The solution of the homogeneous problem (13.17) reveals a critical state in the entire interval $[p_1, p_2]$. Consider a particular solution of system (13.19) which will enable us to establish one more possible critical state.

Assume a particular solution in the form

$$y = a_1 \sin \omega t, \quad z = a_2 \cos \omega t. \tag{13.20}$$

Substituting expressions (13.20) in Eqs. (13.19), we arrive at a non-homogeneous system of algebraic equations for the amplitudes a_1 and a_2

$$-a_1\omega^2 + (p_1^2 - \omega^2)\, a_1 - 2a_2\omega^2 = -g,$$
$$-a_2\omega^2 + (p_2^2 - \omega^2)\, a_2 - 2a_1\omega^2 = -g.$$

Solving the system, we obtain

$$a_1 = -g \frac{p_2^2}{p_1^2 p_2^2 - 2\omega^2 (p_1^2 + p_2^2)},$$
$$a_2 = -g \frac{p_1^2}{p_1^2 p_2^2 - 2\omega^2 (p_1^2 + p_2^2)}.$$

The critical state occurs if the denominator of the above expressions approaches zero; this is possible when

$$\omega^2 = \frac{p_1^2 p_2^2}{2 (p_1^2 + p_2^2)}.$$

If the two stiffnesses are close to each other, we obtain, putting $p_1 = p_2$

$$\omega \simeq \frac{p_1}{2} \simeq \frac{p_2}{2}.$$

Thus, if $k_1 = k_2$ the critical state is possible at an angular velocity equal to half the fundamental critical speed $\omega_{cr}/2$.

14. Effect of Friction

In the above discussion of various factors affecting the motion of an elastic shaft it was assumed that there were no friction forces whatsoever. In actuality these forces invariably accompany any motion and hence the effects of friction must be taken into account if we wish to gain a more penetrating insight into the processes under consideration.

It should be borne in mind that friction far from always reduces the danger of critical states; in some cases it is the friction that may be the cause of critical states of a special kind.

Below we consider successively viscous external resistance, the effect of the oil film in a bearing, dry friction in a bearing and, finally, internal hysteresis in the shaft construction.

Viscous Resistance

There is always some friction present between a rotating disk and the environment. The distributed friction forces can be reduced to the centre of the disk, giving a resultant force T and a resultant couple about the axis of rotation; the latter is balanced by the external torque in steady-state rotation (ω = constant).

When there is an initial eccentricity e, the action of the friction force T leads (in a steady state) to positions of the characteristic points which are shown in Fig. 84: the centre of rotation O, the centre of the shaft section W and the centre of gravity of the disk S are not in a straight line and the segment OW makes an angle γ with the direction OS. The total eccentricity OS is denoted by e_1.

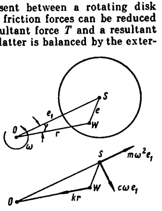


Fig. 84

The disk is acted on by two forces: the elastic reaction of the shaft kr directed from point W to point O and the friction force T which is assumed proportional to the absolute velocity of the centre of the disk ωe_1

$$T = -c\omega e_1, \tag{14.1}$$

where c is the proportionality factor.

The minus sign indicates that the force T is directed oppositely to the velocity of the centre of gravity of the disk. The equations of motion in the direction SO and in a direction perpendicular to it are

$$\left.\begin{aligned} kr\cos\gamma - m\omega^2 e_1 = 0, \\ kr\sin\gamma - c\omega e_1 = 0. \end{aligned}\right\} \tag{14.2}$$

Equations (14.1) and (14.2) involve three unknown quantities deflection r, eccentricity e_1 and angle γ. The angle γ can be expressed by means of Eqs. (14.2) as

$$\tan\gamma = \frac{c}{m\omega},$$

i.e., the angle the disk is ahead of the shaft increases with increase in the coefficient of friction c; in a system with no friction, when $c = 0$, all three points, O, W and S, lie in the same line (Fig. 76).

To make the solution of the problem complete we have to set up one more equation; this equation is of a purely geometric nature and expresses the relation between the elements of triangle OWS

$$e^2 = e_1^2 + r^2 - 2re_1 \cos \gamma.$$

Substituting

$$r = \frac{m\omega^2 e_1}{k \cos \gamma} = \frac{\omega^2 e_1}{p^2 \cos \gamma} \tag{14.3}$$

[this follows from the first equation of (14.2)], we obtain an equation defining e_1

$$e_1 = \frac{ep^2 \cos \gamma}{\sqrt{\omega^4 - 2\omega^2 p^2 \cos^2 \gamma + p^4 \cos^2 \gamma}} . \tag{14.4}$$

According to expression (14.3)

$$r = \frac{e\omega^2}{\sqrt{\omega^4 - 2\omega^2 p^2 \cos^2 \gamma + p^4 \cos^2 \gamma}} .$$

In contrast to the case considered above, the quantities e_1 and r *remain bounded* for any angular velocity ω. The maximum value of the deflection r is reached at $\omega = p$ and is given by

$$r_{\max} = \frac{e}{\sin \gamma} .$$

The maximum value of the eccentricity corresponds to a slightly different relation $\omega = p \cos \gamma$ and is equal to

$$e_{1\max} = \frac{e}{\sin \gamma} .$$

This type of friction limits the deflection of the shaft and hence somewhat reduces the danger of the critical state; in practice friction has no effect on the value of the critical speed if the maximum deflection r of the shaft is regarded as the criterion for a critical state. The self-centring of the disk is not affected by friction, as is seen from expression (14.4).

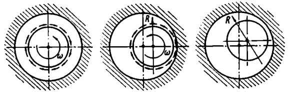
Effect of Oil Film in a Bearing

The mobility of the lubricant in a bearing affects the motion of the shaft journal in the bearing; this may cause dangerous critical states under operating conditions.

Analysing the phenomena associated with the motion of an oil film, we restrict ourselves to a schematic consideration of the main characteristics of forces which arise under these conditions.

When the shaft journal rotates in a bearing, the oil film is partly set in motion and forms a closed stream in the annular cavity between the journal and bearing. The inner surface of the film is then loaded by friction forces acting in the direction of rotation; at the same time the surface of the journal is loaded by equal forces but acting in a direction opposite to the basic rotation. These forces are indicated in Fig. 85*a*.

Suppose that, due to some disturbance, the shaft journal is shifted away from the position corresponding to the steady state (Fig. 85*b*).



(*a*) (*b*) (*c*)

Fig. 85

The configuration of the cavity and the velocities of the particles of the film are then changed. Where the cavity is widened (at the left in Fig. 85*b*) the velocity of the annular stream decreases while on the opposite side (at the right in Fig.85*b*) where the cavity is narrowed the stream velocity increases.

Let us see how this will affect the friction forces. A decrease in the stream velocity means an increase in the relative velocity between the journal and the film. Consequently, the friction forces in the left-hand portion increase, i.e., their increments are positive. Reasoning in the same manner, we may conclude that the friction forces on the opposite side decrease, i.e., their increments are negative; these increments are shown by arrows in Fig. 85*b*. It is seen from the diagram that the resultant R of these increments is perpendicular to the direction of the shift (Fig. 85*c*).

This is the main feature of the phenomenon: any lateral shift of the shaft journal gives rise to an additional friction force directed not opposite but *perpendicular to the shift*.

To a first approximation, the force R may be assumed proportional to the shift, but, being always perpendicular to the shift, it is not a restoring force capable of bringing the shaft journal back to its position. On the contrary, carrying the journal away in its direction, this force will move the journal to a new (also shifted) position. The direction of the force R will then change, i.e., the centre of the journal will eventually rotate about the undisturbed position; the direction of this rotation coincides with that of the basic rotation (Fig. 85c).

Thus, if only these forces are taken into account, the undisturbed motion is unstable. Actually the stability can be maintained owing to viscous-damping forces.

To investigate the combined action of these forces we use a fixed co-ordinate system yz (Fig. 86). The origin A is placed on the line of centres of the bearings; point O is the instantaneous position of the centre of the journal. Point S determines the instantaneous position of the centre of gravity of the disk (this is also the centre W of the section of the shaft at which the disk is located); its co-ordinates are denoted by y_1 and z_1.

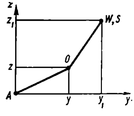


Fig. 86

Consider a disturbed motion of the disk mounted on the rotating shaft; we assume that the additional reactions of the bearings are defined by the law

$$R_y = -\alpha z, \quad R_z = -\alpha y, \qquad (14.5)$$

where α is the factor of proportionality between the additional friction force and the shift.*

We assume that the motion involves a dissipative force proportional to the absolute velocity of the centre of the disk; the components of this force on the co-ordinate axes are $-c\dot{y}$ and $-c\dot{z}$. The differential equations of motion for the *shaft-disk system* are then

$$\left.\begin{aligned} -c\dot{y}_1 - \alpha z &= m\ddot{y}_1, \\ -c\dot{z}_1 + \alpha y &= m\ddot{z}_1, \end{aligned}\right\} \qquad (14.6)$$

and the equations of motion of the disk are

$$\left.\begin{aligned} -k(y_1 - y) - c\dot{y}_1 &= m\ddot{y}_1, \\ -k(z_1 - z) - c\dot{z}_1 &= m\ddot{z}_1. \end{aligned}\right\} \qquad (14.7)$$

In setting up Eqs. (14.6) account is taken of the fact that the shaft-disk system is acted on by the dissipative forces and the components of the additional friction force, R_y and R_z. Equations (14.7) relate to the disk; the forces acting on it are the elastic forces of the shaft and the dissipative forces.

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* This factor in turn is approximately proportional to the cube of the radius of the journal and the square of the angular velocity and inversely proportional to the radial clearance.

Equating the left-hand sides of Eqs. (14.6) and (14.7), we express the co-ordinates of the centre of the journal y and z in terms of the co-ordinates of the centre of the disk y_1 and z_1

$$y = \frac{k^2}{k^2+\alpha^2} y_1 - \frac{k\alpha}{k^2+\alpha^2} z_1,$$

$$z = \frac{k\alpha}{k^2+\alpha^2} y_1 + \frac{k^2}{k^2+\alpha^2} z_1.$$

From expressions (14.6) or (14.7) we can now obtain equations for the co-ordinates y_1 and z_1, which do not contain the co-ordinates y and z

$$-c\dot{y}_1 - \frac{k\alpha^2}{k^2+\alpha^2} y_1 - \frac{k^2\alpha}{k^2+\alpha^2} z_1 = m\ddot{y}_1,$$

$$-c\dot{z}_1 + \frac{k^2\alpha}{k^2+\alpha^2} y_1 - \frac{k\alpha^2}{k^2+\alpha^2} z_1 = m\ddot{z}_1.$$

Introducing the notation

$$\frac{k\alpha^2}{m(k^2+\alpha^2)} = a, \quad \frac{k^2\alpha}{m(k^2+\alpha^2)} = b, \quad \frac{c}{m} = 2n, \tag{14.8}$$

we obtain the system of equations

$$\left.\begin{aligned} \ddot{y}_1 + ay_1 + bz_1 + 2n\dot{y}_1 = 0, \\ \ddot{z}_1 + az_1 - by_1 + 2n\dot{z}_1 = 0. \end{aligned}\right\} \tag{14.9}$$

The particular solution is assumed in a form similar to (13.17)

$$y_1 = Ae^{\lambda t}, \quad z_1 = Be^{\lambda t}. \tag{14.10}$$

Before completing the determination of the characteristic exponent λ, we note that, in general, λ is a complex number

$$\lambda = a_1 + ib_1,$$

so that

$$e^{\lambda t} = e^{a_1 t} e^{ib_1 t}.$$

The second factor involving an imaginary exponent can be represented in terms of trigonometric functions and hence remains bounded at any value of t. The criterion for stability of motion depends on the first factor; if $a_1 < 0$, the motion of the disk is damped out, and if $a_1 > 0$, the disk moves away from the original undisturbed position.*

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* Actually, these relations are not valid at a considerable distance from the undisturbed position as the bearing construction admits only of small shifts. Therefore, the critical state does not imply, of course, an indefinite growth of displacements, but nevertheless it is certainly dangerous.

Thus, the stability of motion is determined by the condition

$$a_1 < 0.$$

Substituting expressions (14.10) in Eqs. (14.9), we obtain a system of homogeneous equations in A and B

$$A(\lambda^2+2n\lambda+a)+Bb=0,$$

$$-Ab+B(\lambda^2+2n\lambda+a)=0.$$

The condition of non-zero solutions for A and B

$$\begin{vmatrix} \lambda^2+2n\lambda+a & b \\ -b & \lambda^2+2n\lambda+a \end{vmatrix}=0$$

leads to the equation

$$(\lambda^2+2n\lambda+a)^2=-b^2. \qquad (14.11)$$

Taking the square root of both sides, we obtain

$$\lambda^2+2n\lambda+a\mp ib=0,$$

whence

$$\lambda=-n\pm\sqrt{n^2-a\pm ib}.$$

If the real part of at least one of the roots λ is greater than zero, the motion is unstable. To separate the real part, we represent the root as

$$\sqrt{n^2-a\pm ib}=\sqrt{\frac{1}{2}\left[\sqrt{(a-n^2)^2+b^2}-a+n^2\right]}\pm$$

$$\pm i\sqrt{\frac{1}{2}\left[\sqrt{(a-n^2)^2+b^2}+a-n^2\right]}.$$

The real part of expression (14.11) is

$$a_1=-n\pm\sqrt{\frac{1}{2}\left[\sqrt{(a-n^2)^2+b^2}-a+n^2\right]}.$$

The stability will be ensured if both values of a_1 are negative, which is possible in the case

$$n>\sqrt{\frac{1}{2}\left[\sqrt{(a-n^2)^2+b^2}-a+n^2\right]}; \qquad (14.12)$$

hence we obtain

$$n^2>\frac{b^2}{4a}.$$

Using notation (14.8), this stability condition may be rewritten as

$$c>k\sqrt{\frac{km}{k^2+\alpha^2}}. \qquad (14.13)$$

When there is no damping ($c = 0$), the motion is *unstable*. For the motion to be stable, the coefficient c must have a certain minimum value; the larger the mass of the disk and the stiffness of the shaft, the greater is this value.

The results obtained seem at first glance to be independent of the angular velocity of the disk. Actually α depends on the angular velocity ω and is proportional to its square. Hence, from expression (14.13) it may be concluded that the higher the speed of the disk, the more easily can the vibration be damped out; in contrast, at low speeds the coefficient c must be large.

Dry Friction in a Bearing

Violent vibrations of a rotating shaft may be set up by dry-friction forces developed between the journal and bearing due to an excessive clearance and deficient lubrication.

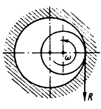

Fig. 87

Figure 87 shows that the friction force R applied to the shaft journal is perpendicular to the shift; in this sense it is similar to the force developed in the presence of an oil film (Fig. 85*b*). Hence, the shaft journal will also roll round the bearing contour, but in this case the centre of the journal will move in a clockwise direction, i.e., *in a direction opposite to the basic rotation*.

Hysteresis Properties of a Shaft Construction

Consider the vibration of a disk mounted at the middle of a rotating vertical hinge-ended shaft of circular section; we assume that the disk is completely balanced on the shaft. The consideration of internal friction in the shaft material will reveal that situations may arise when the straight-line form of the shaft axis becomes unstable and the system starts moving away from the undisturbed position after an arbitrarily small initial disturbance.

Relate the system being considered to a fixed set of coordinate axes xyz, choosing the x axis along a line through the centres of the bearings.

Assume that the centre of the disk is deflected in one way or another from the axis of rotation Ox at a certain instant from which the time is measured and the system is then left alone (Fig. 88*a*). Consider the subsequent process of motion, assuming that the shaft undergoes only bending but no torsion; moreover, we assume that the angular velocity ω of the shaft remains constant at all times.

Denote by $v = v(t)$ and $w = w(t)$ the components of the deflection of the middle of the shaft in the direction of the y and z axes, respectively. Imagine the disk to be separated from the shaft and replace the action of the disk on the shaft by forces P_1 and P_2 (Fig. 88b).

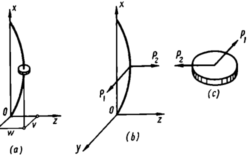

Fig. 88

The forces acting on the disk are shown in Fig. 88c; these forces are equal in magnitude and opposite in sense to the forces P_1 and P_2. The differential equations of motion of the disk are

$$m\ddot{v} = -P_1, \quad m\ddot{w} = -P_2. \tag{14.14}$$

Consider the bending of the shaft in order to relate the forces P_1 and P_2 to the deflections of the shaft v and w.

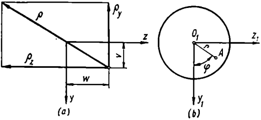

Fig. 89

Denote by ρ the radius of curvature of the shaft axis at the middle section. The projections of ρ on the y and z axes are denoted by ρ_y and ρ_z, respectively. Clearly, the corresponding curvatures are proportional to the deflections (Fig. 89a)

$$\frac{1}{\rho_y} = av, \quad \frac{1}{\rho_z} = aw, \tag{14.15}$$

where a is the proportionality factor which is the same for both directions and depends on the end conditions and the length of the shaft (thus, for a simply supported shaft $a = 12/l^2$).

Consider now the middle section of the shaft (Fig. 89*b*) and attach to the centroid O_1 of this section a moving set of axes y_1, z_1 which always remain parallel to the fixed axes y, z. A point A distant r from the centroid of the section, has the co-ordinates

$$y_1 = r\cos\varphi, \quad z_1 = r\sin\varphi \tag{14.16}$$

with $\varphi = \omega t$.

The extension at point A due to the bending of the shaft is

$$\varepsilon = -\frac{y_1}{\rho_y} - \frac{z_1}{\rho_z}.$$

Substituting expressions (14.15) and (14.16), we obtain

$$\varepsilon = -ar\,(v\cos\omega t + w\sin\omega t).$$

According to (8.46), the normal stress at point A is

$$\sigma = E\varepsilon + c\varepsilon = -Ear\,(v\cos\omega t + w\sin\omega t) -$$

$$-car\,(\dot{v}\cos\omega t - v w\sin\omega t + \dot{w}\sin\omega t + w\omega\cos\omega t).$$

Returning to the co-ordinates y_1, z_1, we have

$$\sigma = -ay_1\,(Ev + c\dot{v} + c\omega w) - az_1\,(Ew + c\dot{w} - c\omega v).$$

The system of elementary forces $\sigma\,dA$ forms the bending moments with respect to the axes y_1, z_1

$$\left.\begin{aligned} M_1 &= -\int_{(A)} \sigma z_1\,dA = aI\,(Ew + c\dot{w} - c\omega v), \\ M_2 &= -\int_{(A)} \sigma y_1\,dA = aI\,(Ev + c\dot{v} + c\omega w). \end{aligned}\right\} \tag{14.17}$$

Referring now to Fig. 88*b*, we note that the bending moment M_1 is produced by the force P_2 and the bending moment M_2 is produced by the force P_1

$$M_1 = \frac{P_2 l}{4}, \quad M_2 = \frac{P_1 l}{4}. \tag{14.18}$$

Equating the right-hand sides of equalities (14.17) and (14.18), we express the forces P_1 and P_2 in terms of the deflections of the centre of gravity of the disk

$$Ew + c\dot{w} - c\omega v = \frac{P_2 l^3}{48I},$$

$$Ev + c\dot{v} + c\omega w = \frac{P_1 l^3}{48I}$$

(it is assumed here that $a = 12/l^2$). From this we obtain the desired expressions for the forces P_1 and P_2

$$\left.\begin{aligned} P_1 &= \frac{48I}{l^3}\,(Ev + c\dot{v} + c\omega w), \\ P_2 &= \frac{48I}{l^3}\,(Ew + c\dot{w} - c\omega v). \end{aligned}\right\} \qquad (14.19)$$

It should be noted that the forces P_1 and P_2 depend not only on the displacements v and w themselves but also on the velocities $\dot{v}$ and $\dot{w}$. Substituting expressions (14.19) in Eqs. (14.14), we obtain the differential equations of motion of the centre of gravity of the disk

$$\left.\begin{aligned} \ddot{v} + p^2\left(v + \frac{c}{E}\,\dot{v} + \frac{c}{E}\,\omega w\right) &= 0, \\ \ddot{w} + p^2\left(w + \frac{c}{E}\,\dot{w} - \frac{c}{E}\,\omega v\right) &= 0, \end{aligned}\right\} \qquad (14.20)$$

where $p^2 = 48\,EI/ml^3$ is the square of the natural frequency of vibration of the disk on a non-rotating perfectly elastic shaft.

To solve Eqs. (14.20) we assume, similarly to expressions (14.10),

$$v = a_1 e^{\lambda t}, \quad w = a_2 e^{\lambda t};$$

we then obtain

$$\begin{aligned} \left(\lambda^2 + p^2\,\frac{c}{E}\,\lambda + p^2\right) a_1 + \frac{cp^2}{E}\,\omega a_2 &= 0, \\ -\frac{cp^2}{E}\,\omega a_1 + \left(\lambda^2 + p^2\,\frac{c}{E}\,\lambda + p^2\right) a_2 &= 0. \end{aligned}$$

The condition of non-zero solutions for a_1 and a_2 is

$$\begin{vmatrix} \lambda^2 + p^2\,\frac{c}{E}\,\lambda + p^2 & \frac{cp^2\omega}{E} \\ -\frac{cp^2}{E}\,\omega & \lambda^2 + p^2\,\frac{c}{E}\,\lambda + p^2 \end{vmatrix} = 0,$$

giving an equation of the fourth degree in λ

$$\left(\lambda^2 + p^2\,\frac{c}{E}\,\lambda + p^2\right)^2 = -\left(\frac{cp^2\omega}{E}\right)^2.$$

This equation is identical in form with Eq. (14.11) which was obtained for the problem of combined effect of an oil film and external viscous friction. Hence, the stability condition can immediately be written in the form of formula (14.12), substituting $a = p^2$, $b = cp^2\omega/E$, $n = cp^2/2E$ in formula (14.12). We then obtain a simple inequality

$$\omega < p. \qquad (14.21)$$

Thus, as long as the speed of the shaft is less than the critical value ($\omega_{cr} = p$), inequality (14.21) is fulfilled and the rotation is *stable*; for $\omega > p$, i.e., in the postcritical range, the rotation of the shaft is *unstable*.

By virtue of external-friction forces, not taken account of here, the stability can be maintained in the postcritical range if, of course, these forces are sufficiently large.

15. Automatic Balancing of Rotating Shafts

To eliminate the deflection produced by the rotation of an unbalanced shaft, special devices are sometimes employed which ensure automatic balancing. This balancing is particularly necessary in cases where appreciable changes may occur in the amount of unbalance of a shaft or rotor under operating conditions; an example is provided by some types of centrifuge in which the symmetry of mass distribution about the axis of rotation may be considerably upset in the loaded condition.

Automatic balancing aids in maintaining the *straight-line form of the shaft axis*, this making it different from self-centring of a disk at high speeds of rotation (in the latter case the mass of the disk is centred *when the shaft is deflected*).

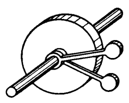

Fig. 90

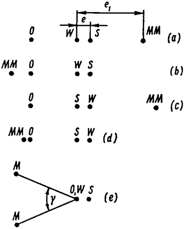

Fig. 91

One type of automatic balancer is shown in Fig. 90. Here the shaft-disk system is complicated by two pendulums which may rotate freely on the shaft. We consider only steady rotation and, for simplicity, neglect gravity effects and inelastic resistances.

Let O (Fig. 91) be a point on a line through the centres of the bearings, W the centre of the shaft section, S the centre of gravity of the disk, M, M the mass centres of the pendulums, $e_1 = \overline{WM}$ the length of the pendulum, $e = \overline{WS}$ the eccentricity.

When there are no pendulums, two mutual arrangements of points O, W and S are possible (see Fig. 76). In either arrangement, the centrifugal force and the elastic force of the shaft act along the same line; hence, adding pendulums, it may be assumed that the axes of both pendulums coincide with this line in either arrangement.

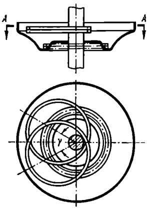

Fig. 92

This leads to four different relative positions of the characteristic points. The cases a and b (see Fig. 91) correspond to the arrangement given in Fig. 76a when the centre of gravity S of the disk is farther from the axis of rotation than is the centre of the shaft section W; these cases differ in the relative position of points M, M. The cases c and d correspond to the arrangement given in Fig. 76b when the centre of gravity S of the disk is closer to the axis of rotation than is the centre of the shaft section W.

These four arrangements exhaust all the possible fundamentally different cases of relative positions of points O, W, S, M, M if they all lie in the same line. However, one more arrangement is possible (see Fig. 91e) which corresponds to a perfectly balanced shaft when the centre of the shaft section W coincides with the centre of rotation of the system O. In this arrangement, there are no elastic forces (since the shaft is not deflected) and the centrifugal force of the disk is balanced by the centrifugal forces of the pendulums; the axes of the pendulums form an angle γ corresponding to a given eccentricity of the disk.

Although the equilibrium is possible in each of the above-enumerated cases of steady rotation, not all these states are stable. A theoretical analysis and experiments show that when $\omega > \omega_{cr}$ only the fifth state is stable. Therefore, in the *postcritical range* such pendulums serve as automatic balancers and prevent the shaft axis from bending; if the eccentricity increases during rotation (point S in Fig. 91 moves to the right), the pendulums draw closer together and the angle γ decreases by an amount necessary to balance the increased centrifugal force of the disk.

In the *subcritical range*, for $\omega < \omega_{cr}$, a stable state is observed in the case a in which the pendulums increase the deflection of the shaft and hence do only harm. Therefore, precautions are taken to "shut off" the pendulums in balancers in the subcritical range.

Figure 92 shows a sketch of an automatic balancer used in washing machines. Here encased rings serve as pendulums. When $\omega < \omega_{cr}$, the centrifugal forces acting on the rings are small, the rings rest on the case bottom and the balancer is "shut off". At $\omega = \omega_{cr}$ the centrifugal forces are sufficient to make the rings "come up" and to bring the balancer into operation.

In some designs of grinding machines encased balls serve as pendulums.

16. Critical States of Helicopter Rotors

The formulas derived for a shaft with a single disk cannot be used if the rotating disk carries masses capable of moving with respect to the disk; thus, the combined mass of the disk and added masses cannot be substituted for m in formula (13.2) for the critical speed.

Among the systems of this type is, for example, a horizontal helicopter rotor consisting of a hub and blades which are attached to the hub by means of vertical hinges. Figure 93 shows a three-blade rotor, with the centre of the hub O and the centres of the vertical hinges A, B, C. Suppose that the helicopter rests on the ground and the centre of the hub O is elastically restrained in the horizontal plane; this elasticity is provided by the whole construction of the helicopter (from the hub centre to the points of contact of the helicopter wheels with the ground).

We schematize the mass properties of the system and assume that the blades are completely balanced, the mass m_* of each blade being concentrated at a distance b from the centre of the corresponding vertical hinge. Assume also that the hub is not completely balanced and its centre of gravity G is at a distance e from the hub centre O on the bisector of the angle BOC.*

Due to the unbalance of the system the rotation of the rotor gives rise to a centrifugal force which produces an additional elastic displacement r of the hub centre, as represented in Fig. 93*b*. Notation: O' = displaced position of hub centre, G' = its centre of gravity, A', B', C' = centres of vertical hinges. Of course, these letters designate an instantaneous position of the rotor; in time, the points O', G', A', B' and C' describe circles with centre at point O, which determines the axis of rotation of the system. It is important to note that the axes of the blades suspended in hinges B' and C' will no longer be along lines $O'B'$ and $O'C'$ as the centrifugal forces of the blades must pass through the centre of rotation O. The angle that the axis of either blade makes with line $O'A'$ is somewhat less than 60°; denote it by $60° - \varepsilon$ (Fig. 93c).

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* This type of unbalance of the rotor is obviously a special case, a fact which will not, however, affect the final conclusions. At the same time this scheme is the easiest for analysis.

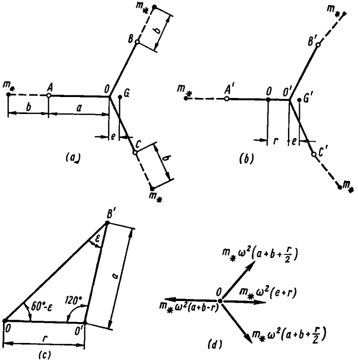

Fig. 93

From $\Delta OO'B'$ we have

$$\frac{r}{\sin\varepsilon} = \frac{a}{\sin(60°-\varepsilon)} = \frac{\overline{OB'}}{\sin 120°};$$

hence we obtain approximately

$$\sin\varepsilon \simeq \varepsilon \simeq \frac{\sqrt{3}}{2}\,\frac{r}{a},$$

$$\overline{OB'} \simeq a + \frac{r}{2}.$$

It is now possible to determine the centrifugal forces for the blades:

for blade A

$$m_*\omega^2(a+b-r),$$

for blade B

$$m_*\omega^2\left(a+b+\frac{r}{2}\right),$$

for blade C

$$m_*\omega^2\left(a+b+\frac{r}{2}\right).$$

The centrifugal force diagram is shown in Fig. 93*d*. In addition to the centrifugal forces of the blades, it includes the centrifugal force of the hub $m\omega^2(e+r)$, where m is the mass of the hub.

The sum of all these forces is directed along line OO' and is equal to

$$I=2m_*\omega^2\left(a+b+\frac{r}{2}\right)\cos(60^\circ-\varepsilon)+m\omega^2(e+r)-\\-m_*\omega^2(a+b-r).$$

Substituting

$$\cos(60^\circ-\varepsilon)\cong\cos 60^\circ+\varepsilon\sin 60^\circ=\frac{1}{2}+\frac{3}{4}\,\frac{r}{a},$$

we find

$$I=m\omega^2(r+e)+3m_*\omega^2 r\left(1+\frac{b}{2a}\right). \tag{16.1}$$

Another important relation exists between the quantities I and r: the displacement r is equal to the quotient of the force I and the spring stiffness k of the elastic system, $r=I/k$. Substituting expression (16.1), we obtain a simple equation for determining the displacement r; solving it, we find

$$r=\frac{e}{\frac{k}{m\omega^2}-1-3\frac{m_*}{m}\left(1+\frac{b}{2a}\right)}.$$

From this it is immediately seen that the critical speed is equal to

$$\omega_{cr}^*=\sqrt{\frac{k}{m+3m_*\left(1+\frac{b}{2a}\right)}}. \tag{16.2}$$

Note the additional term $b/2a$ appearing in the denominator of the last expression. It is this term that expresses the effect of the mobility of the blades relative to the hub; if the system is imagined with no vertical hinges (i.e., rigid rotor), the critical speed of the system would be expressed by the formula $\omega_{cr}=\sqrt{\frac{k}{m+3m_*}}$, i.e., it would be considerably higher.

Formula (16.2) holds true for any number n of blades (if $n \geqslant 3$), but the factor 3 should be replaced by the number n.

Without investigating the properties of a two-blade rotor, we only note that the mobility of the blades has a still greater effect here: instability occurs over a range of speeds (between the value ω_{cr}^* taking account of the mobility of the blades and the value ω_{cr} for a rigid rotor).

17. Shaft with Several Disks. Rigid Rotor on Elastic Supports

Shaft with Several Disks

Consider the critical states of shafts carrying several disks (Fig. 94) and determine the critical speeds from the conditions of elastic equilibrium of the deflected shaft subjected to centrifugal forces $m_1\omega_{cr}^2 r_1$, $m_2\omega_{cr}^2 r_2$, . . ., $m_n\omega_{cr}^2 r_n$, where $m_1, m_2, \ldots, m_n$ are the masses of the disks, $r_1, r_2, \ldots, r_n$ are the corresponding elastic deflections of the shaft. Using the influence coefficients, we write the expressions for the deflection in the general form (gyroscopic effects are neglected for the sake of simplicity)

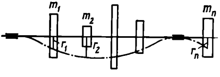

Fig. 94

$$\left.\begin{aligned}
r_1 &= m_1\omega_{cr}^2 r_1\delta_{11} + m_2\omega_{cr}^2 r_2\delta_{12} + \ldots + m_n\omega_{cr}^2 r_n\delta_{1n},\\
r_2 &= m_1\omega_{cr}^2 r_1\delta_{21} + m_2\omega_{cr}^2 r_2\delta_{22} + \ldots + m_n\omega_{cr}^2 r_n\delta_{2n},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
r_n &= m_1\omega_{cr}^2 r_1\delta_{n1} + m_2\omega_{cr}^2 r_2\delta_{n2} + \ldots + m_n\omega_{cr}^2 r_n\delta_{nn}.
\end{aligned}\right\} \qquad (17.1)$$

This homogeneous system of equations admits of non-zero solutions for $r_1, r_2, \ldots, r_n$ if the determinant of the coefficients of the system is zero

$$\begin{vmatrix}
m_1\omega_{cr}^2\delta_{11} - 1 & m_2\omega_{cr}^2\delta_{12} & \ldots & m_n\omega_{cr}^2\delta_{1n}\\
m_1\omega_{cr}^2\delta_{21} & m_2\omega_{cr}^2\delta_{22} - 1 & \ldots & m_n\omega_{cr}^2\delta_{2n}\\
\cdots & \cdots & \cdots & \cdots\\
m_1\omega_{cr}^2\delta_{n1} & m_2\omega_{cr}^2\delta_{n2} & \ldots & m_n\omega_{cr}^2\delta_{nn} - 1
\end{vmatrix} = 0. \qquad (17.2)$$

Equation (17.2) is identical to Eq. (7.31) derived previously as a condition for determining the natural frequencies of transverse vibration of the same system with no rotation. Consequently, the critical speeds of a multidisk shaft are equal to the natural frequencies of flexural vibration of the same shaft with no rotation. This conclusion, which is also valid for shafts with distributed mass, is a generalization of the result obtained by considering a shaft with a single disk. Consequently, ω_{cr} may be determined by the methods developed for linear systems of several degrees of freedom. To each of the critical speeds corresponds a particular shape of the deflection curve of the shaft coinciding with one of the natural modes of flexural vibration.

Rigid Rotor on Elastic Supports

The scheme of an elastic shaft with several heavy disks (see above) has become classical in the analysis of critical speeds of many real structures. However, in cases where the stiffness of a rotor is large in comparison with the stiffness of supports (as in some types of loom spindle) the scheme of an elastically supported absolutely rigid rotor is more appropriate.

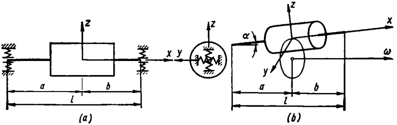

Fig. 95

In considering this scheme (Fig. 95*a*) we assume the spring stiffnesses of the supports k_1 and k_2 to be different. The rotor is assumed to be completely balanced; its moments of inertia about the x, y and z axes through the centre of gravity are J_x, J_y, J_z, respectively.

To determine the critical speed we consider the rotor in a displaced position (Fig. 95*b*). Here, the left end of the rotor describes a circle of radius r about its undisturbed position while the rotor axis is deflected from the original position by an angle α and describes a conic surface.

As pointed out above, in the critical state the system of centrifugal forces and elastic reactions is in equilibrium regardless of the scale of the deflections. Hence, in this case we can write

$$m\omega_{cr}^2 (r+\alpha a) = k_1 r + k_2 (r+\alpha l), \tag{17.3}$$

where $r + \alpha a$ = radial displacement of centre of gravity,
$m\omega_{cr}^2 (r + \alpha a)$ = centrifugal force of rotor,
$r + \alpha l$ = radial displacement of right end of rotor,
$k_1 r$, $k_2 (r + \alpha l)$ = reactions of supports.

Besides, during the motion the rotor is subjected to the moment of the reactions of supports about the y axis

$$M_y = -k_1 ra + k_2 (r+\alpha l)\, b \tag{17.4}$$

Its value must satisfy Euler's equation of motion

$$J_y \dot{\omega}_y + (J_x - J_z)\, \omega_z \omega_x = M_y,$$

where ω_x, ω_y, ω_z are the projections of the angular velocity ω on the x, y and z axes attached to the rotor.

According to Fig. 95*b*

$$\omega_x = \omega_{cr}, \quad \omega_y = 0, \qquad \omega_z = -\omega_{cr}\alpha.$$

Euler's equation then yields

$$M_y = -(J_x - J_z)\, \omega_{cr}^2 \alpha. \tag{17.5}$$

In the particular case of a very short rotor, which may be regarded as a disk, $J_x = 2J_z$ and formula (17.5) reduces to expression (13.9).

Equating expressions (17.4) and (17.5), we obtain

$$-k_1 ra + k_2 (r+\alpha l)\, b = -(J_x - J_z)\, \omega_{cr}^2 \alpha. \tag{17.6}$$

The system of equations (17.3) and (17.6) is homogeneous in the displacements r and α. Rewrite this system as

$$r (m\omega_{cr}^2 - k_1 - k_2) + \alpha\, (am\omega_{cr}^2 - k_2 l) = 0,$$
$$-r (k_1 a - k_2 b) + \alpha\, [(J_x - J_z)\, \omega_{cr}^2 + k_2 bl] = 0.$$

In order to have r and α different from zero, the determinant of the coefficients of this system must be zero

$$\begin{vmatrix} m\omega_{cr}^2 - k_1 - k_2 & am\omega_{cr}^2 - k_2 l \\ -(k_1 a - k_2 b) & (J_x - J_z)\, \omega_{cr}^2 + k_2 bl \end{vmatrix} = 0.$$

Expanding the determinant, we obtain a biquadratic equation

$$\omega_{cr}^4 + \omega_{cr}^2 \left(\frac{k_1 a^2 + k_2 b^2}{J_x - J_z} - \frac{k_1 + k_2}{m} \right) + \frac{k_1 k_2 l^2}{m\,(J_x - J_z)} = 0.$$

from which the critical speed may be found.

As a simple example let us consider vibrations produced by a unit *step force*, i.e., by a force $P = 1$ suddenly applied at time $t = \tau$ and then remaining constant at all times (Fig. 97*a*).

For $t \geqslant \tau$ the differential equation becomes

$$\ddot{x} + p^2 x = \frac{1}{m} \tag{18.3}$$

The solution of the equation must satisfy the initial conditions

$$x = 0 \quad \text{and} \quad \dot{x} = 0 \quad \text{at} \quad t = \tau. \tag{18.4}$$

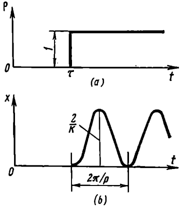

Fig. 97

This solution represents the sum made up of the solution of the corresponding homogeneous differential equation

$$x_1 = C_1 \sin pt + C_2 \cos pt$$

and a particular solution of the given differential equation (18.3)

$$x_2 = \frac{1}{mp^2} = \frac{1}{k},$$

i.e.,

$$x = C_1 \sin pt + C_2 \cos pt + \frac{1}{k}.$$

After determining the constants from conditions (18.4)

$$C_1 = -\frac{\sin p\tau}{k}, \quad C_2 = -\frac{\cos p\tau}{k},$$

we find

$$x = \frac{1 - \cos p(t - \tau)}{k}. \tag{18.5}$$

This law of motion is illustrated in Fig. 97*b*. As is seen, the maximum value of x is $x_{max} = 2/k$, i.e., *twice* the displacement caused by a statically applied force $P = 1$.

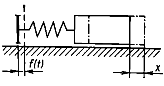

Fig. 98

The problem that involves forced vibrations induced *kinematically* (i.e., by "kinematic disturbance") can be reduced to the same standard form (18.2). To explain this, consider again the same single-mass system but assume that the vibration is due to a given vibration of the point of attachment of the spring (Fig. 98); suppose that the law of motion of this point is given as $f(t)$. At the current instant the extension of the spring is $x - f$ and the mass is acted on by the spring force $-k(x-f)$; accordingly the differential equation of motion is

$$-k(x-f)=m\ddot{x}$$

or

$$\ddot{x}+p^2x=\frac{kf(t)}{m}.$$

The product $kf(t)$ may be taken as the effective driving force

$$P(t)=kf(t); \tag{18.6}$$

this leads again to the standard form (18.2).

General Solution of Standard Equation

Method of variation of arbitrary constants. The solution of the non-homogeneous equation (18.2) should be sought as the sum of the solution of the corresponding equation with right-hand side equal to zero (i.e., free-vibration equation) and some particular solution of the given equation (18.2). Instead of choosing a particular solution corresponding to the given right-hand side in each specific case, it is better to use the general method of variation of arbitrary constants. This will make it possible to obtain a result suitable for any laws of variation of the driving force.

The idea of this method is that a particular solution of the given equation (18.2) is sought in the form

$$x=C_1\sin pt+C_2\cos pt \tag{18.7}$$

corresponding to the homogeneous equation. In this case, however, the quantities C_1 and C_2 should be considered variable rather than constant. Thus, the problem of determining the function $x(t)$ is replaced by the problem of determining two functions, $C_1(t)$ and $C_2(t)$. Since we have only one equation (18.7) at our disposal, the functions C_1 and C_2 may be related by one more arbitrary expression.

Set up the expression for the velocity

$$\dot{x}=C_1p\cos pt-C_2p\sin pt+\dot{C}_1\sin pt+\dot{C}_2\cos pt$$

and relate $\dot{C}_1$ and $\dot{C}_2$ by

$$\dot{C}_1\sin pt+\dot{C}_2\cos pt=0. \tag{18.8}$$

The velocity is then written in a simpler form

$$\dot{x}=C_1p\cos pt-C_2p\sin pt.$$

Find now the acceleration

$$\ddot{x}=-C_1p^2\sin pt-C_2p^2\cos pt+\dot{C}_1p\cos pt-\dot{C}_2p\sin pt. \tag{18.9}$$

Substituting expressions (18.7) and (18.9) in Eq. (18.2) we obtain

$$\dot{C}_1 \cos pt - \dot{C}_2 \sin pt = \frac{P(t)}{mp} \tag{18.10}$$

From Eqs. (18.8) and (18.10) we can find the derivatives

$$\dot{C}_1 = \frac{P(t)}{mp} \cos pt, \qquad \dot{C}_2 = -\frac{P(t)}{mp} \sin pt.$$

Integrating gives

$$\left.\begin{aligned} C_1 &= \frac{1}{mp} \int_0^t P(\tau) \cos p\tau \, d\tau + B_1, \\ C_2 &= -\frac{1}{mp} \int_0^t P(\tau) \sin p\tau \, d\tau + B_2, \end{aligned}\right\} \tag{18.11}$$

where B_1 and B_2^0 are constants.

The symbol τ is introduced to distinguish the time varying during integration (from zero to t) from the upper limit t, which should be considered constant in this process.

Substituting expressions (18.11) in Eq. (18.7), we obtain the general solution of the given equation (18.2)

$$x = \frac{1}{mp} \left[\sin pt \int_0^t P(\tau) \cos p\tau \, d\tau - \cos pt \int_0^t P(\tau) \sin p\tau \, d\tau \right] + \\ + B_1 \sin pt + B_2 \cos pt. \tag{18.12}$$

Putting $\sin pt$ and $\cos pt$ under the integral signs and combining the latter, we obtain the general solution of the problem

$$x = B_1 \sin pt + B_2 \cos pt + \frac{1}{mp} \int_0^t P(\tau) \sin p(t-\tau) \, d\tau.$$

Accordingly we obtain for the velocity

$$\dot{x} = B_1 p \cos pt - B_2 p \sin pt + \frac{1}{m} \int_0^t P(\tau) \cos p(t-\tau) \, d\tau. \tag{18.13}$$

The values of the constants B_1 and B_2 can be determined only after the initial conditions of the motion are indicated. If $x = x_0$ and $\dot{x} = v_0$ at $t = 0$, we can find from expressions (18.12) and (18.13)

$$B_1 = \frac{v_0}{p}, \quad B_2 = x_0.$$

The solution then becomes

$$x = x_0 \cos pt + \frac{v_0}{p} \sin pt + \frac{1}{mp} \int_0^t P(\tau) \sin p(t-\tau)\, d\tau.$$

Here, the first two terms express the *free vibration* induced by the initial disturbances x_0 and v_0, and the last term expresses the *forced vibration* produced by the driving force.

In the case of zero initial conditions when the motion is started with $x_0 = 0$ and $v_0 = 0$, we obtain

$$x = \frac{1}{mp} \int_0^t P(\tau) \sin p(t-\tau)\, d\tau; \tag{18.14}$$

this basic formula will be used in what follows.

Below we will also need an alternate form of the solution which is obtained by integrating solution (18.14) by parts.

We set

$$\sin p(t-\tau)\, d\tau = dw, \quad P(\tau) = u,$$

then

$$w = \frac{\cos p(t-\tau)}{p}, \quad du = \dot{P}(\tau)\, d\tau.$$

By the formula for integration by parts we obtain, making the substitution $mp^2 = k$,

$$x = \frac{1}{mp} \int_0^t P(\tau) \sin p(t-\tau)\, d\tau = \frac{1}{mp} \Big[\frac{P(\tau)}{p} \cos p(t-\tau) - {} - \frac{1}{p} \int_0^{\tau} \dot{P}(\tau) \cos p(t-\tau)\, d\tau \Big]_0^t = \frac{P(t) - P(0) \cos pt}{k} - {} - \frac{1}{k} \int_0^t \dot{P}(\tau) \cos p(t-\tau)\, d\tau. \tag{18.15}$$

If the force $P(0) = 0$ at the initial instant, the solution takes the form

$$x = x_{st} - \frac{1}{k} \int_0^t \dot{P}(\tau) \cos p(t-\tau)\, d\tau, \tag{18.16}$$

where $x_{st} = P(t)/k$ is the variable "static" displacement calculated on the assumption that there are no inertia forces.

Derivation of general solution by superposing the effects of elementary impulses. In order to gain a clearer insight into the physical significance of expression (18.14), consider first the free vibration of an elastically mounted mass subjected to a unit momentary impulse $S = 1$; let this impulse be applied at time τ.

Immediately after the impulse disappears, the displacement of the mass is still zero: $x(\tau) = 0$. However, on the basis of the principle of impulse and momentum the mass acquires instantaneously a velocity $v(\tau) = 1/m$ due to the application of the impulse.

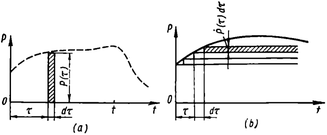

Fig. 99

These formulas determine the initial conditions of free vibrations which will take place at $t > \tau$. Use these conditions; to do this, we substitute τ for t on the right-hand sides of the expressions for displacement and velocity

$$x = a\sin(pt + \alpha), \qquad v = \dot{x} = ap\cos(pt + \alpha)$$

and the values indicated by the initial conditions, on the left-hand sides. We then obtain

$$0 = a\sin(p\tau + \alpha),$$
$$\frac{1}{m} = ap\cos(p\tau + \alpha).$$

From these relations we find

$$a = \frac{1}{mp}, \quad \alpha = -p\tau.$$

Consequently the motion is governed by the law

$$x = \frac{1}{mp}\sin p(t - \tau). \tag{18.17}$$

After solving this auxiliary problem we revert to the case of a driving force and consider it as a succession of infinitesimal impulses $P(\tau)\,d\tau$ (Fig. 99*a*). According to formula (18.17) the displacement at time $t > \tau$ due to such an impulse is $\frac{P(\tau)\,d\tau}{mp}\sin p(t-\tau)$. The displacement caused by the whole succession of impulses over the interval $(0, t)$ is found by integration; this leads again to formula (18.14).

We emphasize that the summation of the results is legitimate due to the linearity of the system; in the case of a non-linear system this approach would be unjustified.

When a system undergoes a series of momentary impacts * (their impulses are denoted by $S_1, S_2, \ldots$) at times $\tau_1, \tau_2, \ldots$, the integration should be replaced by summation, i.e.,

$$y = \frac{1}{mp}\sum_{i=1}^{n} S_i \sin p(t-\tau_i),$$

where n is the number of the last impulse preceding the time t.

Derivation of general solution by superposing the effects of elementary step forces. An arbitrarily assigned force $P(\tau)$ can be represented as the sum of an infinite sequence of step forces (Fig. 99*b*).

We first isolate the initial step force corresponding to the initial value of the force, $P(0)$; according to expression (18.5) its action is given by

$$x_1 = \frac{P(0) - P(0)\cos pt}{k}. \tag{18.18}$$

The subsequent variation of the given force is represented as a succession of infinitesimal step forces $\dot{P}(\tau)\,d\tau$ each of which produces a motion defined by the same expression (18.5) multiplied by the value of the step force $\dot{P}(\tau)\,d\tau$.

The overall effect of the whole succession of elementary step functions is determined by the integral

$$x_2 = \int_0^t \frac{\dot{P}(\tau)}{k}[1-\cos p(t-\tau)]\,d\tau = \frac{P(t)-P(0)}{k} - \\ -\frac{1}{k}\int_0^t \dot{P}(\tau)\cos p(t-\tau)\,d\tau. \tag{18.19}$$

Adding expressions (18.18) and (18.19) together, we finally find the previous result (18.15) which was obtained earlier by purely formal transformations.

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* Here we mean "massless" impacts whose impulses are preassigned and assumed to be independent of the motion of the system itself.

Kinematic Disturbance

Revert to the case shown in Fig. 98 when forced vibration results from the motion of the point of attachment of an elastic constraint. By relation (18.6), when this point vibrates according to a law $f(t)$, the mass vibrates as if it were subjected to a driving force $P(t) = kf(t)$.

This relation enables one to use all the results obtained above. The basic solution (18.14) is rewritten as

$$x = p\int_0^t f(\tau)\sin p(t-\tau)\,d\tau.$$

Similarly, in place of formula (18.15) we obtain [for $f(0) = 0$]

$$x = f(t) - \int_0^t \dot{f}(\tau)\cos p(t-\tau)\,d\tau. \qquad (18.20)$$

Some Cases of Aperiodic Disturbance

Effect of a linearly increasing force (Fig. 100*a*). We use expression (18.16) putting $\dot{P}(t) = a$

$$x = x_{st} - \frac{a}{kp}\sin pt.$$

The graph of the motion is shown in Fig. 100*b*. The displacements build up according to a complex law expressed by the sum of a sine function and a linear function. The higher the rate of increase of the force P (i.e., the greater is a), the more significant is the additional sinusoidal vibration.

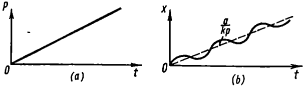

Fig. 100

Vibration of a sprung mass when moving over an irregular road (Fig. 101). Assume that the road profile is given by the equation

$$z = h(1 - e^{-\nu x}),$$

where h is the limit approached by the height of the profile, γ is a parameter characterizing the curvature of the profile.

Denote by W the weight of the mass and by v its horizontal velocity and choose the origin of time at the instant the supporting point passes the beginning of the irregularity. Then $x = vt$ and the motion of the supporting point along a vertical is governed by the law

$$f(t) = h(1 - e^{-\gamma vt}).$$

Differentiating gives

$$\dot{f}(t) = \gamma vhe^{-\gamma vt}.$$

Using formula (18.20), we obtain the law of motion of the mass along a vertical

$$z = f(t) - \int_0^t \gamma vhe^{-\tau v\tau} \cos p(t-\tau)\, d\tau.$$

Since it is not the absolute change in the position of the mass that is of interest but its vibration with respect to the supporting point,

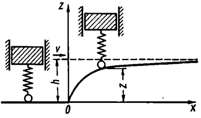

Fig. 101

we consider the difference giving an additional deformation of the spring

$$z_* = z(t) - f(t) = -\int_0^t \gamma vhe^{-\gamma v\tau} \cos p(t-\tau)\, d\tau.$$

Integrating yields

$$z_* = h\cos^2\alpha \left[e^{-\gamma vt} + \frac{\sin(pt-\alpha)}{\sin\alpha}\right],$$

where α is defined by the relation

$$\tan\alpha = \frac{p}{\gamma v}.$$

It is seen that for very small velocities the parameter α approaches $\pi/2$ and the difference $z - f$ approaches zero. On the other hand, for very high velocities (and also for very large values of the parameter γ) the parameter α approaches zero and the vibration is approximately defined by the law

$$z_* = h\left(e^{-\gamma v t} - \cos pt\right).$$

A further analysis will enable one to find the maximum value of the difference z_*, acceleration, etc.

Effect of slowly varying forces. Consider the foregoing solution (18.16) to the problem of forced vibration. The first term represents

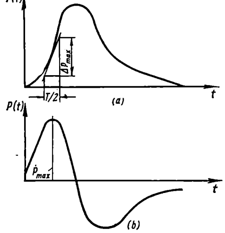

Fig. 102

the static deflection caused by the force $P(t)$. The second term of this formula gives a correction to the static deflection. As is seen, the correction depends on the rate of change of the force, $\dot{P}(t)$.

For small rates of increase of the external load the dynamic correction to the static solution is relatively small and the loading may be considered as practically static.

Below is given an estimate for the dynamic correction in the general case of an exciting force. If the curve $P(t)$ has one maximum (Fig. 102*a*), then, denoting the maximum value of $\dot{P}(t)$ by $\dot{P}_{max}$

(Fig. 102*b*), we have

$$\left| -\int_0^t \dot{P}(\tau)\cos p(t-\tau)\,d\tau \right| < \frac{\dot{P}_{\max}T}{2}.$$

The product $\dot{P}_{\max}\ T/2$ represents the greatest possible increment of the exciting force during a time interval equal to one half-period of free vibration; denoting this product by $\Delta P_{\max}$ (Fig. 102), we obtain

$$x < \frac{P+\Delta P_{\max}}{k}.$$

If the force increases uniformly during the time t_0, then $\dot{P} = P_{\max}/t_0$ and the dynamic addition is $\Delta P_{\max} = P_{\max}T/2t_0$. Its relative value is

$$\frac{\Delta P_{\max}}{P_{\max}} = \frac{T}{2t_0}. \tag{18.21}$$

Hence, if the period of free vibration is small in comparison with the length of action of the force, it may be considered to be slowly varying and dynamic effects negligible, i.e., the force may be assumed to be statically applied.

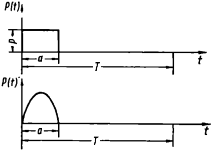

Fig. 103

Effect of rapidly vanishing forces. Let us now consider cases where an exciting force acts during a very short interval of time. Even a very large load may be safe if its length of action is small in comparison with the period of free vibration of the system.

Consider the action of a force P which is suddenly applied at time $t = 0$, acts during a time interval a and then vanishes as suddenly (Fig. 103). It may be shown that if $a < T/2$ the maximum deflection of the system is reached after the force is removed. In this case, according to solution (18.14) we have for $t > a$

$$x = \frac{P}{mp} \int_0^a \sin p(t-\tau)\, d\tau = \frac{2P}{k} \sin \frac{pa}{2} \sin p\left(t - \frac{a}{2}\right). \quad (18.22)$$

Denote the ratio of the time interval a to the period of free vibration T by α; then

$$\frac{pa}{2} = \frac{\pi a}{T} = \pi\alpha.$$

According to expression (18.22) the maximum deflection is

$$x_{\max} = 2x_{st} \sin \pi\alpha.$$

Consequently, the magnification factor is

$$\mu = \frac{x_{\max}}{x_{st}} = 2 \sin \pi\alpha.$$

For forces of short duration the magnification factor has the following values:

α	0	0.01	0.02	0.03	0.05	0.10	0.15	0.25	0.5
μ	0	0.062	0.126	0.188	0.313	0.618	0.908	1.413	2.000

It is seen that if a force acts for a small fraction of the period of free vibration, the effect of this short-time force is many times smaller than its static effect. A similar conclusion may be drawn in cases where an exciting force is represented by one half-wave of a sine curve (see Fig. 103), etc.

We will show that the action of a short-time force may be approximated by its impulse. For $t > a$ we have the solution in the form

$$x = \frac{1}{mp} \int_0^a P(\tau) \sin p(t-\tau)\, d\tau$$

or

$$x = \frac{1}{mp}\left[\sin pt \int_0^a P(\tau) \cos p\tau\, d\tau - \cos pt \int_0^a P(\tau) \sin p\tau\, d\tau\right] =$$

$$= \frac{1}{mp}\left[\sin pt \int_0^a P(\tau) \cos \frac{2\pi\tau}{T}\, d\tau - \cos pt \int_0^a P(\tau) \sin \frac{2\pi\tau}{T}\, d\tau\right].$$

But since the ratio τ/T is less than the ratio a/T, τ/T is a small number. Hence we can write

$$x \cong \frac{\sin pt}{mp} \int_0^a P(\tau)\, d\tau.$$

The integral here involved is the impulse of the force $P(t)$; denoting it by S, we obtain $x \cong \frac{S}{mp} \sin pt$, i.e., the motion of the system is determined by the magnitude of the impulse of a short-time force. As is seen, the details of variation of the force during the time interval a are immaterial; the governing factor is the impulse S.

Effect of a Harmonic Force

The case when a driving force varies according to a harmonic law

$$P = P_0 \sin \omega t \tag{18.23}$$

is one of the most common in practice. In expression (18.23), P_0 is the amplitude of the force, ω its frequency.

The vibratory process produced by this force under zero initial conditions can be described by means of formula (18.14)

$$x = \frac{P_0}{mp} \int_0^t \sin \omega t \sin p(t-\tau)\, d\tau. \tag{18.24}$$

Evaluating the integral, we find (for $\omega \neq p$)

$$x = \frac{P_0}{m(p^2-\omega^2)} \left(\sin \omega t - \frac{\omega}{p} \sin pt \right). \tag{18.25}$$

Substituting $mp^2 = k$ and denoting P_0/k by x_{st} (the deflection caused by a statically applied constant force P_0), we obtain

$$x = \frac{x_{st}}{1 - \frac{\omega^2}{p^2}} \left(\sin \omega t - \frac{\omega}{p} \sin pt \right). \tag{18.26}$$

An inspection of solution (18.26) indicates that the vibrations set up under zero initial conditions are of a complex nature; they involve two parts: vibration with the frequency ω of the driving force (first term) and vibration with the natural frequency p (second term).

The first vibration is usually called a *forced* vibration, and the second a *free* vibration. It should be borne in mind that this terminology is only a matter of convenience. The point is that the second vibration is also produced by the given driving force and its amplitude depends on the same force; in this sense the second vibration should also be regarded as a forced vibration. However, the above names are in common use because only the first term has the frequency of the disturbance whereas the second term varies with the natural frequency of the system.

Note that the neglect of inelastic resistances obscures a very important fact: in practice the second term (free vibration) is damped out in the course of time whereas the first term maintains a constant

The integral here involved is the impulse of the force $P(t)$; denoting it by S, we obtain $x \cong \frac{S}{mp} \sin pt$, i.e., the motion of the system is determined by the magnitude of the impulse of a short-time force. As is seen, the details of variation of the force during the time interval a are immaterial; the governing factor is the impulse S.

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Note that the neglect of inelastic resistances obscures a very important fact: in practice the second term (free vibration) is damped out in the course of time whereas the first term maintains a constant amplitude. This provides an additional justification for the second term to be called a free vibration.

The process of adding two vibrations with different frequencies ω and p is shown in Fig. 104. Figure 104*a* pertains to the case when $\omega > p$, and Fig. 104*b* to the case when $\omega < p$. Both cases are characterized by a rapid decay of free vibration. Therefore, the consideration may be restricted to the *steady-state*, undamped part of the solution

$$x = \frac{x_{st}}{1 - \frac{\omega^2}{p^2}} \sin \omega t. \tag{18.27}$$

As is seen, the forced vibration is performed with the frequency of the driving force which makes, as it were, the system follow its

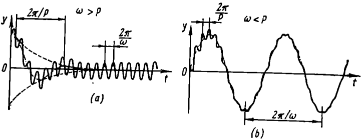


Fig. 104

law of variation. The amplitude of forced vibration

$$a = \frac{x_{st}}{\left|1 - \frac{\omega^2}{p^2}\right|} \tag{18.28}$$

differs from the deflection x_{st} calculated on the assumption of static action of the force P_0. The ratio a/x_{st} may be called the magnification factor

$$\mu = \frac{1}{\left|1 - \frac{\omega^2}{p^2}\right|}. \tag{18.29}$$

The magnification factor depends only on the frequency ratio ω/p. A curve showing the relation of the magnification factor μ to ω/p is given in Fig. 105*a*.

For low frequencies of a driving force the magnification factor is close to unity. The magnification factor increases rapidly with increasing frequency ω and becomes infinite at $\omega = p$. This corresponds to the *resonant condition* when the amplitude of forced vibration builds up without limit (if inelastic resisting forces are taken into account the amplitude at resonance is bounded but is usually so large that the resonant condition should be considered dangerous anyway).

If the frequency ω is higher than the frequency p, the amplitudes become finite; for $\omega/p > \sqrt{2}$ the magnification factor becomes less than unity, i.e., the dynamic effect is less significant than the corresponding static effect. For very large values of the ratio ω/p the magnification factor becomes very small. This means that a force of high frequency produces no perceptible vibration in a low-frequency elastic system; the latter "has no time" to respond to very rapid variations of the driving force.

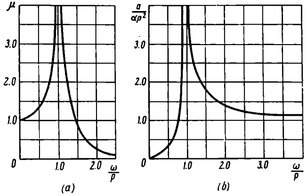


Fig. 105

It was assumed above that the amplitude of the driving force was not related to its frequency. However, more often than not the reverse is true. For instance, in the case of an unbalanced rotor the driving force transmitted to the supports is

$$P = m_0\omega^2 e \sin \omega t,$$

where m_0 = mass of rotor,
e = its eccentricity,
ω = speed.

In this case the amplitude of the driving force $m_0\omega^2 e$ is proportional to the square of the frequency ω^2.

In such cases, in place of solution (18.25) we have to use (for $\omega \neq p$)

$$x=\frac{m_0 e\omega^2}{m\left(p^2-\omega^2\right)}\left(\sin \omega t-\frac{\omega}{p} \sin p t\right).$$

The amplitude of steady-state vibration is then given by the expression

$$a=\frac{\alpha p^2}{\left|\frac{p^2}{\omega^2}-1\right|}$$

in which the parameter of the system

$$\alpha=\frac{m_0 e}{m}$$

is independent of the frequency ω.

Figure 105*b* shows the variation of the amplitude of vibration with the ratio ω/p. As is seen, at $\omega = p$ resonance occurs and for $\omega \gg p$ the amplitude approaches a value αp^2.

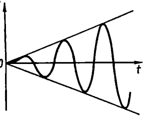


Fig. 106

Consider in greater detail the case when the frequencies coincide $\omega = p$ (resonance). Integral (18.24) assumes the form

$$x=\frac{P_0}{mp} \int_0^t \sin p\tau \sin p(t-\tau)\, d\tau .$$

After evaluating we obtain

$$x=x_{st}(\sin pt-pt \cos pt).$$

The graph of the motion is shown in Fig. 106. As is seen, when the frequencies coincide the amplitude increases according to a linear law and does not become infinite in a finite interval of time. This provides a means for passing through resonance in practice as the equality $\omega = p$ is fulfilled only for one instant during the speeding-up period and the resonant amplitudes may not build up to dangerous values.

Example 18. An unbalanced engine weighing 4 tons and running at $n = 800$ rpm is mounted at the middle of a simply supported beam of span 5 m. Determine the magnification factor if the beam is of rolled steel No. 30*a* I-section ($I = 8{,}950$ cm^4).

We find the natural frequency of the system assuming $E = 2.1 \times 10^6$ kgf/cm^2

$$p=\sqrt{\frac{48EI}{ml^3}}=\sqrt{\frac{48\times 2.1\times 10^6\times 8{,}950\times 981}{4{,}000\times 500^3}}=42.1 \text{ sec}^{-1}.$$

The frequency of the driving force is

$$\omega=\frac{\pi n}{30}=\frac{3.14\times 800}{30}=83.8 \text{ sec}^{-1}.$$

The magnification factor

$$\mu=\frac{1}{\left|1-\frac{\omega^2}{p^2}\right|}=\frac{1}{\left|1-\frac{83.8^2}{42.1^2}\right|}=0.338$$

is considerably smaller than unity. Note that replacing this beam by a stiffer one may impair and not improve the performance of the structure. Thus, if No. 30*a* I-beam is replaced by No. 45*a* I-beam ($I = 35{,}280$ cm^4), the natural frequency increases to a value of 83.7 sec^{-1} and the magnification factor as calculated by formula (18.29) will increase hundreds of times.

Example 19. Determine the dynamic stresses at the middle section of the beam considered in Example 18 if the centrifugal force developed by the engine is 1,000 kgf.

The bending moment produced by the static load is

$$M_{st}=\frac{Pl}{4}=\frac{4{,}000\times 500}{4}=500{,}000 \text{ kgf-cm}.$$

The maximum bending moment produced by the dynamic load is

$$M_d=0.338\,\frac{1{,}000\times 500}{4}=42{,}200 \text{ kgf-cm}.$$

The total bending moment is

$$M=500{,}000+42{,}200=542{,}200 \text{ kgf-cm}.$$

The bending stress is

$$\sigma=\frac{M}{W}=\frac{542{,}200}{597}=907 \text{ kgf/cm}^2.$$

Example 20. An engine of weight 2.4 tons is mounted on ten identical springs of diameter $D = 12$ cm. The diameter of the section of the spring coil is $d = 3$ cm; the shear modulus of the spring material is $G = 0.8\times 10^6$ kgf/cm^2, $n = 800$ rpm. Determine the required number of coils of the spring if the magnification factor of the installation is to be 0.2.

From the condition

$$\frac{1}{\left|1-\frac{\omega^2}{p^2}\right|}=0.2$$

we find

$$\frac{\omega}{p}=2.45.$$

Since

$$\omega=\frac{\pi n}{30}=\frac{3.14\times 800}{30}=83.8 \text{ sec}^{-1},$$

the required value of the natural frequency is

$$p=\frac{83.8}{2.45}=34.2 \text{ sec}^{-1}.$$

Consequently, the following equality must be fulfilled

$$\sqrt{\frac{k}{m}}=34.2\ \text{sec}^{-1}.$$

Substituting $m=2{,}400:981=2.44$ kgf-sec²/cm, we find the required stiffness of all springs

$$k=34.2^2\times 2.44=2{,}850\ \text{kgf/cm}.$$

The stiffness of one spring is

$$k_1=\frac{2{,}850}{10}=285\ \text{kgf/cm}.$$

Using Table 1 we find

$$k_1=\frac{Gd^4}{8D^3n}=285\ \text{kgf/cm},$$

i.e.,

$$\frac{0.8\times 10^6\times 3^4}{8\times 12^3 n}=285;$$

hence $n=16.5$.

It is necessary to take at least 17 coils as the stiffness of the system and the magnification factor decrease with the number of coils. If $n<16.5$ is taken, the magnification factor will be larger than specified by the conditions of the problem.

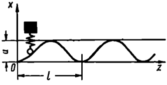


Fig. 107

Consider now the case of kinematic disturbance. As stated above, the amplitude of an equivalent driving force is calculated by multiplying the amplitude of impressed vibration by the spring stiffness of the constraint. Thus, if the amplitude of *harmonic vibration of the foundation* is A, the amplitude of an equivalent driving force is $P_0=kA$. Hence, $x_{st}=P_0/k=A$.

According to formula (18.28), the amplitude of vibration of the system is

$$a=\frac{A}{\left|1-\frac{\omega^2}{p^2}\right|}. \tag{18.30}$$

Example 21. Determine the speed of an automobile at which the rear suspension is in resonance. The road profile is given by the law (Fig. 107)

$$x=\frac{a}{2}\left(1-\cos\frac{2\pi z}{l}\right),$$

where a = depth of trough,

l = wavelength (take $l=7$ m).

The static load on the rear spring is $G=715$ kgf, the stiffness of the spring is 2,250 kgf/m. Assume the vibration of the rear part of the automobile to be independent of the vibration of the front part and neglect the elasticity of the tires.

We find the natural frequency

$$p=\sqrt{\frac{kg}{G}}=\sqrt{\frac{2{,}250\times 9.81}{715}}=5.56\ \text{sec}^{-1}.$$

The vertical vibration of the wheel follows the law

$$f=\frac{a}{2}\left(1-\cos\frac{2\pi vt}{7}\right);$$

the frequency is

$$\omega=\frac{2\pi v}{7}\ \text{sec}^{-1}.$$

Equating this quantity to the natural frequency

$$\frac{2\pi}{v}=5.56\ \text{sec}^{-1},$$

we find the resonant speed

$$v=\frac{5.56\times 7}{2\times 3.14}=6.2\ \text{m/sec}.$$

With a further increase in the speed, the magnification factor decreases rapidly. For instance, at $v=25$ m/sec

$$\mu=\frac{1}{\left|1-\left(\frac{2\pi\times 25}{7\times 5.56}\right)^2\right|}=0.066,$$

i.e., the amplitude of vibration of the body is only 6.6 per cent of the half-depth of the trough.

In connection with this example let us consider the question of smooth running of an automobile.

The natural criterion for smooth running seems at first glance to be the vertical acceleration. However, when the body is vibrating according to the law

$$x_1=a_1\sin\omega_1 t$$

and according to the law

$$x_2=a_2\sin\omega_2 t$$

the physiological sensations may be different even if the maximum accelerations are the same, $a_1\omega_1^2=a_2\omega_2^2$.

It is not only the magnitude of the maximum acceleration that matters but also the rate of change of acceleration; for the same maximum acceleration, the vibration with abrupt changes in acceleration, i.e., with a higher frequency, is more unpleasant. Therefore, it is sometimes believed that the product of the maximum acceleration and the frequency of vibration $a\omega^3$ should be chosen as a measure of smoothness of running. This quantity represents the maximum value of the third derivative $\dddot{x}_{\max}$ and is sometimes called the *jerk*.

Application of the Theory of Forced Vibration to Measuring Techniques

Instruments used for registering and recording dynamic processes in mechanical systems operate on the principle following from the theory of forced vibration.

Frequency meters serve to determine experimentally the frequencies of vibrating objects. Figure 108*a* shows a sketch of a reed frequency meter based on the resonant principle. The instrument contains a number of thin plates built into the frame, with small additional masses attached to their ends. The lengths of these plates are different, and so are their natural frequencies, which can, of course, be predetermined by calculation. The frame of the frequency meter is attached to a vibrating object so that each of the plates is subjected to vibrations induced by kinematic disturbance. The plate whose natural frequency is close to the frequency of vibration being measured is forced into a state of particularly violent vibration; in this way the desired frequency of vibration is determined.

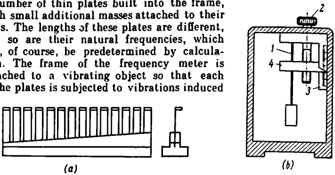


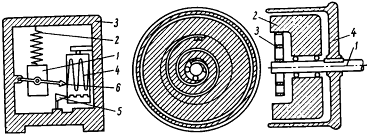
Fig. 108

The difference of the natural frequencies of two adjacent strips is 0.5 cps so that the measurement error cannot exceed this value. The useful frequency range of the instrument is not wide, between 10 and 20 cps.

The principal part of the frequency meter shown in Fig. 108*b* is an elastic bar *1* with a weight at its end. After the instrument is mounted on a vibrating object the free length of the bar and consequently its natural frequency are varied gradually by moving plank *4* smoothly by means of screw *2*. When the vibration of the bar becomes very violent, the plank is stopped and the frequency of vibration determined from a calibrated scale *3*.

Vibrographs of the inertia (seismic) type are used to record displacements of points of vibrating objects or to record angles of rotation (in the latter case they are known as *torsiographs*). The frame of the instrument is rigidly attached to the object whose vibration is measured; the instrument incorporates a practically fixed body in relation to which the displacements of the frame, i.e., virtually the displacements of the vibrating object are noted.

A sketch of a simple vibrograph with mechanical measurement of vibration is shown in Fig. 109*a*. The major part of the vibrograph is a heavy weight *1* (*seismic mass*) suspended in frame *3* by a soft elastic spring *2*. The frame of the vibrograph is attached to the structure whose vibration is studied and it vibrates with the latter.



(*a*) (*b*)

Fig. 109

The weight-spring system is also subjected to vibrations induced by kinematic disturbance. The natural frequency of this system is very low also because of the small stiffness of the spring; therefore the ratio ω/p is usually very large and, according to formula (18.28), the amplitude of vibration of the weight is only a negligible fraction of the amplitude of vibration of the frame of the instrument so that in practice weight *1* may be considered fixed.

The instrument is mounted so as to make the axis of the rotating drum coincident with the direction of the vibration under investigation. Stylus *6* attached to weight *1* will then trace the vibration curve on a strip of paper or film moving with drum *4*. A timing device automatically marks intervals of time on the strip (time marker is designated by the figure *5*).

A linkage system serves to record vibrations to an enlarged scale; this permits measurement of vibrations with an amplitude of 0.01 mm and larger.

In order to have the error of the vibrograph sufficiently small, the natural frequency of the weight-spring system must be considerably lower than the frequency to be measured. Moreover, for precise operation of the vibrograph its mass must be much smaller than the mass of the vibrating structure, otherwise the mounting of the vibrograph will change the vibratory properties of the system under investigation (this is also true for frequency meters).

The same principle is used in *torsiographs*—instruments for recording torsional vibrations (Fig. 109*b*). A heavy flywheel *2* fits freely on axle *1* of the instrument. The flywheel and the axle of the instrument are connected together by a flexible spiral spring *3* so that the natural frequency of torsional vibration is very low. Pulley *4* is keyed on the instrument axle and connected to the shaft whose torsional vibration is studied.

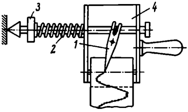


Fig. 110

As the shaft rotates, the instrument also rotates but, because of the flexibility of the spring, high-frequency torsional vibrations are not practically transmitted to the flywheel and it rotates uniformly with the average angular velocity of the shaft. The vibration of the instrument axle with respect to the flywheel is transmitted to the recording stylus by means of a linkage system.

In view of some disadvantages of the linkage system of the recording device the mechanical linkage is replaced by an electrical system in modern inertia-type instruments. Mechanical vibrations are transformed into electrical vibrations; the latter are measured or recorded (by means of an oscillograph) after proper amplification. However, the basic principle—the presence of an inertia element with an elastic (and usually somewhat damped) suspension — remains in this scheme too.

Vibration pickup. A sketch of a vibration pickup (also called a *hand vibrograph*) is shown in Fig. 110.

While in a stationary vibrograph the stylus is fixed and the frame is movable, giving a motion to the strip, in a hand vibrograph stylus *1* moves following the vibrations of a follower pin *3*; the frame of the instrument *4* is held in place by hand and, in view of the small stiffness of spring *2*, may be considered stationary. Although some displacements of the frame are unavoidable during operation of the vibrograph, these vibrations of the frame can easily be distinguished in the record obtained as they differ markedly from the basic vibrations by their relatively large period.

Spring *2* ensures the tracking motion of the follower pin. For proper operation of the instrument the force P of the compressed spring should hold the pin point in contact with the vibrating object. If the compression of the spring is inadequate, the contact is upset when the vibrating member begins to move in the opposite direction (away from the pin). The acceleration in this motion (in harmonic vibration) is equal to $a\omega^2$, where a is the amplitude of the vibration under investigation; for continuous contact the initial compressive force of the spring P must provide an acceleration of the pin of no less magnitude. Denoting by m the combined mass of the pin and associated parts of the instrument, we arrive at the condition

$$\frac{P}{m} \geqslant a\omega^2$$

or

$$P \geqslant ma\omega^2.$$

Example 22. Determine the required initial compressive force of the spring of a hand vibrograph used for recording vibrations with amplitude up to 0.6 cm and frequency up to 600 sec^{-1}; the combined weight of the pin and moving parts is 0.05 kgf.

The mass of the movable part is

$$m = \frac{0.05}{981}\ \text{kgf-sec}^2\text{cm}^{-1}.$$

The required compressive force of the spring is

$$P \geqslant \frac{0.05}{981}\, 0.6 \times 600^2 = 10.8\ \text{kgf}.$$

Dynamic pressure pickup. Consider the principle of a dynamometric indicator (Fig. 111), which serves to record the pressure in a cylinder of a piston engine. The cylinder *1* of the indicator is connected with the cylinder of the engine; the motion of the piston *2* of the indicator under the action of a variable pressure is recorded (to an enlarged scale) on a moving strip *3*.

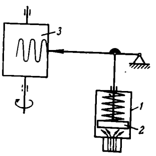


Fig. 111

The condition for precision of the vibrographs and torsiographs described above is a relatively small stiffness of the spring; the condition for proper operation of the indicator is, however, the opposite property, viz. a *large stiffness* of its spring. Indeed, in order that the record of displacement x may be read as the record of pressure q, it is desirable to have a direct proportionality between q and x as if the pressure q were static. As pointed out above, a force may be considered as statically applied if the duration of its increase is long in comparison with the period of free vibration of the piston.

In the case of abrupt changes in pressure this requires a small period of free vibration, i.e., a large stiffness of the spring of the indicator.

If the pressure increases in time according to a linear law, the relative error of readings is given by formula (18.21).

Example 23. Determine the relative error of indicator readings when the following data are given: time required for pressure to reach its maximum value is $t = 0.1$ sec; cross-sectional area of cylinder of indicator $A = 1.6$ cm^2; reduced weight of moving parts of indicator 60 gf. It is known that at a static pressure of 2 atm the end of the pointer of the indicator moves through 0.5 cm; the gear ratio of the linkage system is $i = 4$.

Find the stiffness k of the spring of the indicator. At a pressure of 2 atm the force of the compressed spring is $2 \times 1.6 = 3.2$ kgf and the deformation of the spring is $0.5 : 4 = 0.125$ cm.

Consequently,

$$k = \frac{3.2}{0.125} = 25.6 \text{ kgf/cm}.$$

The mass of moving parts is

$$m = \frac{0.06}{981} = 0.0000612 \text{ kgf-sec}^2\text{/cm}.$$

The natural frequency of vibration of the instrument is

$$p = \sqrt{\frac{k}{m}} = \sqrt{\frac{25.6}{0.0000612}} = 646 \text{ sec}^{-1}.$$

The period of natural vibration is

$$T = \frac{2\pi}{646} = 0.0097 \text{ sec}.$$

The relative error of instrument readings is

$$0.0097 : 2 \times 0.1 = 0.0485,$$

i.e., less than 5 per cent.

Principle of Vibration Isolation

To isolate disturbing sources from adjacent structural elements vibration absorbers are introduced. Protection of the base against the action of driving forces is called *active* isolation (Fig. 112*a*) and protection against kinematic disturbances is called *passive* isolation (Fig. 112*b*). The first type of vibration isolation is employed when mounting various unbalanced engines; the second type serves to protect structures attached to the vibrating base or to protect transportation facilities (automobiles, railroad cars, etc.).

In the theory of active isolation the basic problem is that of determining the force transmitted to the base; in the theory of passive isolation, it is the problem of finding the amplitude of the vibration the object to be protected is forced into. In view of the fact that rational mounting presupposes operation far away from the resonant conditions it is permissible to carry out the basic analysis of vibration isolation without regard to inelastic resisting forces. At the same time it should be borne in mind that these forces must always be taken into account when assessing the danger involved in passing through resonance (in starting or stopping the machine).

The analysis of vibration isolation is required in all cases; the use of isolating devices without preliminary analysis is not permissible.

Active isolation. If the driving force is a harmonic one and the structure is undamped, then, according to formula (18.28), the maximum force transmitted to the base is

$$N = ka = \frac{kx_{st}}{\left|1-\frac{\omega^2}{p^2}\right|} = \frac{P_0}{\left|1-\frac{\omega^2}{p^2}\right|} = \mu P_0.$$

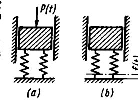


Fig. 112

Hence, the efficiency of active isolation depends on the magnitude of the magnification factor μ to be used. This requires accordingly a low value of the natural frequency p, which may be obtained by reducing the stiffness of the mounting of the machine or by increasing the vibrating mass.

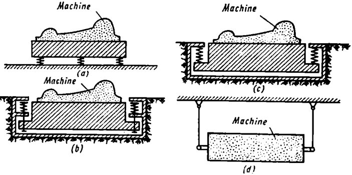


Fig. 113

For this purpose a set of elastic constraints (vibration isolators), usually in the form of steel springs or rubber elements, are introduced under the frame of the machine to be isolated (or under the rigid base to which the machine is attached). Two equally efficient variants of mounting are in use: *supporting variant* when vibration isolators are placed under the base of the machine (Fig. 113*a*) and *suspension* *variant* when vibration isolators are placed above the bottom of the base; in the latter case the vibration isolators may be either in compression (Fig. 113*b*) or in tension (Fig. 113*c*).

If horizontal driving forces prevail in the machine to be isolated a pendulum suspension may also be used to advantage (Fig. 113*d*); a very low value of the natural frequency and an appreciable isolation effect can be obtained by using long hangers.

The suspension should be made soft only in the direction of the driving force (or couple). Thus, the operation of single-phase electric motors brings about a variable torque. Therefore, for vibration isolation the constraint corresponding to rotations of the stator must be flexible; at the same time, if there is a gear wheel on the shaft, the normal operation of the gear drive requires high stiffness for vertical and horizontal displacements.

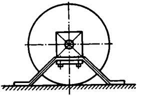


Fig. 114

A rational way of mounting such an electric motor is shown in Fig. 114. The support of the bearing is a steel strip bent so that the axes of the inclined portions intersect on the axis of the shaft. For any vertical or horizontal displacement this support is very stiff (inclined portions are subjected to tension or compression and deform only slightly), whereas for rotations of the stator this support is relatively flexible (inclined portions are subjected to bending and deform easily).

For the same reasons the automobile engine suspension is provided with little stiffness for elastic rotations about a longitudinal axis and considerably greater stiffness in other directions perpendicular to this axis.

It should be noted that vibration isolation of slow-speed machines (when the frequency of the disturbance is not high) may require a very low natural frequency and accordingly impractically great flexibility of vibration absorbers. To overcome this difficulty the vibrating mass is artificially increased in such cases. This serves a twofold purpose: first, the natural frequency is reduced and, second, sufficient stiffness of the system is preserved.

Passive isolation. As pointed out above, passive isolation is used to protect instruments and precision machines against vibrations transmitted from the supporting structure.

The amplitude of vibration of an isolated object is expressed in terms of the amplitude of vibration of the base by formula (18.30). It is apparent that passive isolation should use the same idea of making the mounting soft, as in active isolation. It is generally required that the natural frequency of the isolated object shall not exceed one-fourth of the frequency of vibration of the base.

If the frequency of the disturbance is not fixed, it is necessary to introduce inelastic resistances in the mounting system. Thus, the irregularities of a road may have the shape of a sine curve with the wavelength varying over a wide range. Therefore, there is a real danger that the body of a moving automobile may be in a state of resonance; to limit resonant amplitudes the automobile suspension is always provided with hydraulic shock absorbers which dissipate a considerable amount of energy during vibration (Fig. 115*a*). This absorber system has a disadvantage: it does not afford sufficient comfort of passengers when subjected to shocks which are transmitted to the automobile body with almost no alleviation (pneumatic tires not being considered). To obtain the necessary softness of the suspension it may be provided with an additional flexible element as shown in Fig. 115*b*.

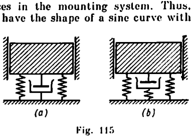


Fig. 115

Effect of Two Harmonic Forces with Close Frequencies; Beats

If a system of one degree of freedom is subjected to two different harmonic driving forces, then, because of linearity, the overall effect can be determined by adding the effects produced by each of the forces acting separately. Thus, a driving force of the form

$$P = P_1 \sin \omega_1 t + P_2 \sin \omega_2 t$$

produces complex vibrations

$$x = a_1 \sin \omega_1 t + a_2 \sin \omega_2 t. \tag{18.31}$$

If the frequencies ω_1 and ω_2 are close to each other, the resultant motion is of a peculiar type known as *beating* (Fig. 116*a*). The beating phenomenon is characterized by a periodic variation of the amplitude.

To analyse this phenomenon we first transform solution (18.31) as

$$x = a_1 \sin \omega_1 t + a_2 \sin \omega_2 t = \frac{a_1 + a_2}{2} [\sin \omega_1 t + \sin \omega_2 t] +$$

$$+ \frac{a_1 - a_2}{2} [\sin \omega_1 t - \sin \omega_2 t] = (a_1 + a_2) \cos \frac{\omega_1 - \omega_2}{2} t \sin \frac{\omega_1 + \omega_2}{2} t +$$

$$+ (a_1 - a_2) \sin \frac{\omega_1 - \omega_2}{2} t \cos \frac{\omega_1 + \omega_2}{2} t.$$

Denoting

$$b_1 = (a_1 + a_2)\cos\frac{\omega_1 - \omega_2}{2}t,$$

$$b_2 = (a_1 - a_2)\sin\frac{\omega_1 - \omega_2}{2}t,$$

we write the preceding formula as

$$x = b_1 \sin\frac{\omega_1 + \omega_2}{2}t + b_2\cos\frac{\omega_1 + \omega_2}{2}t$$

or

$$x = \sqrt{b_1^2 + b_2^2}\sin\left(\frac{\omega_1 + \omega_2}{2}t + \alpha\right),$$

where

$$\alpha = \operatorname{arc\,tan}\frac{b_1}{b_2}.$$

It should be emphasized that, because of closeness of ω_1 and ω_2,

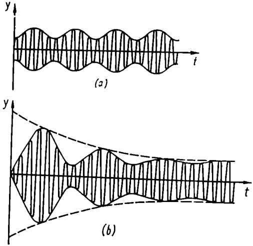


Fig. 116

the quantities b_1, b_2 and α vary slowly. Thus, the motion can be described by the expression

$$x = a\sin(\omega t + \alpha),$$

where $\omega = \frac{\omega_1 + \omega_2}{2}$ = mean value of frequency,

$a = \sqrt{b_1^2 + b_2^2}$ = slowly varying amplitude of vibration,

α = slowly varying phase.

Hence, the motion is almost sinusoidal, the amplitude of vibration a being a slowly varying function of time. The period of variation of the amplitude a (beat period) is

$$T_a = \frac{2\pi}{\omega_1 - \omega_2}\,.$$

Since the difference $\omega_1 - \omega_2$ is small, the period T_a is considerably greater than the period of vibration

$$T = \frac{4\pi}{\omega_1 + \omega_2}\,.$$

Beats may be induced by a single driving force $P_0 \sin \omega t$ in the neighbourhood of resonance when the frequency ω is close to the natural frequency p. From solution (18.25) it is seen that in this case the vibration consists of two harmonics with close frequencies ω and p. Transformations similar to those used above lead to the conclusion that. here too, the resultant vibration is a sinusoidal one with varying amplitude. In this case, however, the process is not steady; the free vibration disappears gradually because of damping and only the forced vibration persists, so that beating will cease (Fig. 116*b*).

Effect of an Arbitrary Periodic Driving Force
(Resolution into Harmonic Components)

In practical applications we often have to deal with periodic driving forces of a more complex form than considered above.

Thus, Fig. 117*a* shows the law of variation of the torque produced by a four-stroke internal-combustion engine. Another example (periodic "massless" shocks) is shown in Fig. 117*b*.

The forces (torques) of the type considered have a pronounced period of vibration T but are not representable by a single analytic expression. It is customary in such cases to expand a periodic force in a Fourier series. The force is represented as a sum of harmonic components and the effect produced by each of the components is determined; then the resulting individual effects are summed up.

A periodic force $P(t)$ can be represented as a Fourier series

$$P(t) = a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots,$$

where $\omega = 2\pi/T$ is the *fundamental frequency* of the disturbance.

The coefficients a and b are evaluated by the well-known formulas

$$a_0 = \frac{1}{T} \int_0^T P(t)\,dt,$$

$$a_1 = \frac{2}{T} \int_0^T P(t) \cos \omega t\,dt, \qquad b_1 = \frac{2}{T} \int_0^T P(t) \sin \omega t\,dt,$$

$$a_2 = \frac{2}{T} \int_0^T P(t) \cos 2\omega t\,dt, \qquad b_2 = \frac{2}{T} \int_0^T P(t) \sin 2\omega t\,dt,$$

. . . .

Based on solution (18.27) for one harmonic, we find

$$x = \frac{a_0}{k} + \frac{a_1 \cos \omega t + b_1 \sin \omega t}{k\left(1 - \frac{\omega^2}{p^2}\right)} + \frac{a_2 \cos 2\omega t + b_2 \sin 2\omega t}{k\left[1 - \left(\frac{2\omega}{p}\right)^2\right]} + \dots \qquad (18.32)$$

This solution is made up of the constant term a_0/k (corresponding to the average value of the driving force) and a number of harmonic

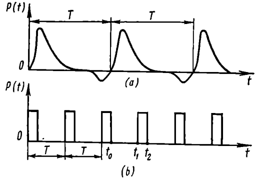


Fig. 117

vibrations with frequencies, ω, 2ω, ... If the natural frequency coincides with the frequency of any one of the harmonics $n\omega$ ($n = 1, 2, \dots$), the corresponding term in formula (18.32) approaches infinity.

Consequently, in the general case of a periodic driving force resonance occurs not only when the natural frequency p is equal to the fundamental frequency ω of the driving force but also when p is a multiple of ω [in some particular cases formula (18.32) does not contain certain terms and resonance does not occur for all multiplicities].

The foregoing method clearly discloses the conditions under which resonance occurs. The disadvantage of this method is that the calculations are too complicated because it is necessary to deal with a very large number of terms in expression (18.32). Thus, the driving force shown in Fig. 117a has to be replaced by about ten harmonics for sufficient accuracy.

Effect of Periodic Impulses (Closed Form of Solution)

Let us investigate the effect of periodic impulses (Fig. 117b) assuming the length of action of each of them to be vanishingly small.

We consider any one of the periods T choosing the origin of time at the end of action of the preceding impulse (for example, at time t_0 in Fig. 117b). Denote the displacement and velocity at the initial instant by x_0 and v_0, respectively.

During the period considered (before the application of the next impulse) the motion is a free vibration with natural frequency p and is defined by the equation

$$x = x_0 \cos pt + \frac{v_0}{p} \sin pt. \tag{18.33}$$

Consequently,

$$v = \dot{x} = -px_0 \sin pt + v_0 \cos pt.$$

At the end of this period, just before the next impulse (time t_1), we obtain

$$x_1 = x_0 \cos pT + \frac{v_0}{p} \sin pT,$$

$$v_1 = -px_0 \sin pT + v_0 \cos pT.$$

As a result of the action of the succeeding impulse the velocity changes suddenly by an amount S/m (where S is the value of the impulse). Therefore, immediately after the next impulse (time t_2)

$$x_2 = x_1 = x_0 \cos pT + \frac{v_0}{p} \sin pT,$$

$$v_2 = v_1 + \frac{S}{m} = -px_0 \sin pT + v_0 \cos pT + \frac{S}{m}.$$

Since the process is periodic, these quantities must be equal to x_0 and v_0

$$x_0 = x_0 \cos pT + \frac{v_0}{p} \sin pT,$$

$$v_0 = -px_0 \sin pT + v_0 \cos pT + \frac{S}{m}.$$

These equations involve two unknown quantities, x_0 and v_0. Solving the equations, we find

$$x_0 = \frac{S}{2mp} \cot \frac{pT}{2},$$

$$v_0 = \frac{S}{2m};$$

the law of motion (18.33) takes the form

$$x = \frac{S}{2mp}\left(\sin pt + \cos pt \cot \frac{\pi p}{\omega}\right) \tag{18.34}$$

(here the substitution $T = 2\pi/\omega$ has been made).

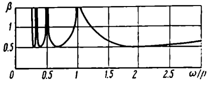


Fig. 118

The simple and closed form of this solution offers a simple way of investigating the effect of periodic shocks; note that in this case the method of resolution into harmonic components would lead again to infinite sums.

The amplitude of vibration is given by

$$a = \frac{S}{2mp}\sqrt{1 + \cot^2 \frac{\pi p}{\omega}} = \frac{S}{2mp\left|\sin \frac{\pi p}{\omega}\right|}.$$

The quotient S/mp is the maximum deflection produced by one impulse; therefore, the expression

$$\beta = \frac{1}{2\left|\sin \frac{\pi p}{\omega}\right|} \tag{18.35}$$

may be called the *repetition factor*; this factor characterizes the increased effect of repeated impulses.

Figure 118 shows the variation of the factor β as a function of the frequency ratio ω/p. From formula (18.35) and Fig. 118 it is seen that when the frequencies are identical or multiple ($p = n\omega$, $n = 1, 2, \ldots$) resonance occurs. The minimum possible value of the repetition factor is 1/2.

Effect of an Arbitrary Periodic Driving Force (Closed Form of Solution)

The foregoing method can be used to obtain a closed solution to the problem involving the action of an arbitrary periodic force.

Let a system be acted on by an arbitrarily given periodic force of period T (Fig. 119). Divide the time axis into intervals of length T (one of the division points is chosen arbitrarily). The law of variation of the force can be written as $P(\tau)$, where τ is the time measured from the beginning of any one of the intervals; in other intervals the law $P(\tau)$ is repeated with the corresponding shift of the time origin.

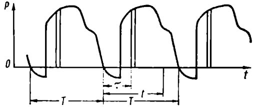


Fig. 119

Isolate an infinite series of identical elementary impulses $P(\tau)\,d\tau$ from the given law of variation of the force. The displacement of the system at time t produced by this series of elementary impulses can be determined by formula (18.34). In this formula, however, by t is meant the time elapsed between the instant a particular impulse is applied and the instant for which the displacement x is determined; in our case this time should be denoted by $t-\tau$ (for $\tau < t$) and by $T+t-\tau$ (for $\tau > t$). Integrating between the corresponding limits, we find the displacement x produced by all the *series* of elementary impulses

$$x(t)=\frac{1}{2mp}\left\{\int_0^t P(\tau)\left[\sin p(t-\tau)+\cos p(t-\tau)\cot\frac{pT}{2}\right]d\tau+\right.$$

$$\left.+\int_t^T P(\tau)\left[\sin p(T+t-\tau)+\cos p(T+t-\tau)\cot\frac{pT}{2}\right]d\tau\right\}.$$

After transformations we finally obtain

$$x(t) = \frac{1}{2mp}\int_0^T P(\tau)\left[\cot\frac{pT}{2}\cos p(t-\tau) - \sin p(t-\tau)\right]d\tau +$$
$$+ \frac{1}{mp}\int_0^t P(\tau)\sin p(t-\tau)\,d\tau. \qquad (18.35a)$$

Let, for example, $P(\tau) = b\tau$ within the limits $(0, T)$ (Fig. 120). Of course, this case of a *saw-tooth* driving force can be handled by the method presented on p. 221; the solution, however, would require consideration of a large number of terms, the number being larger, the greater the accuracy required. In the present case the computations are rather elementary and lead to a closed form of the solution

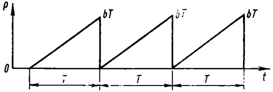


Fig. 120

$$x = \frac{b}{kp}\left[2pt + pT\left(\cos pt - \cot\frac{pT}{2}\sin pt\right)\right].$$

19. Linear Systems of One Degree of Freedom with Inelastic Resisting Forces

Effect of Viscous Resistance

General solution. If viscous resistance is taken into account, the basic equation of forced vibration assumes the form

$$\ddot{x} + 2n\dot{x} + p^2x = \frac{P(t)}{m}. \qquad (19.1)$$

It differs from the equation of damped free vibration (5.4) by the right-hand side and from the equation of forced vibration of a system with no inelastic resistance (18.2) by the second term on the left-hand side. To obtain the general solution we use the method which was applied above to the solution of a similar problem with $n = 0$.

Let a momentary impulse S be applied to a system of one degree of freedom at time τ; the ensuing vibratory process can be described by Eq. (5.5). Determine the constants a and α from the initial conditions of the motion:

$x=0$ and $\dot{x}=S/m$ at $t=\tau$. The first condition yields

$$\alpha=-p_*\tau.$$

From the second condition we find

$$a=\frac{Se^{n\tau}}{mp_*},$$

where p_* is given by formula (5.5*b*).

Thus, the free vibrations induced by the impulse S are governed by the law

$$x_*=\frac{S}{mp_*}e^{-n(t-\tau)}\sin p_*(t-\tau)$$

and are damped out in time.

As before, we consider the driving force $P(\tau)$ as a sequence of infinitesimal impulses $P(\tau)\,d\tau$. The general solution of the problem concerning the action of the force $P(t)$ then becomes

$$x=\frac{1}{mp_*}\int_0^t P(\tau)\,e^{-n(t-\tau)}\sin p_*(t-\tau)\,d\tau. \tag{19.2}$$

This formula is applicable for any law $P(t)$ and reduces to formula (18.14) for $n=0$.

Harmonic driving force. In the important particular case of the action of a harmonic force

$$P=P_0\sin\omega t$$

solution (19.2) yields

$$x=a\sin(\omega t-\gamma), \tag{19.3}$$

where

$$a=\frac{x_{st}}{\sqrt{\left(1-\frac{\omega^2}{p^2}\right)^2+\frac{4\omega^2n^2}{p^4}}}, \tag{19.4}$$

$$\tan\gamma=\frac{2\omega n}{p^2-\omega^2}. \tag{19.5}$$

In the first formula

$$x_{st}=\frac{P_0}{k}=\frac{P_0}{mp^2}.$$

Introduce, as before, the *magnification factor*

$$\mu=\frac{a}{x_{st}}=\frac{1}{\sqrt{\left(1-\frac{\omega^2}{p^2}\right)^2+\frac{4\omega^2n^2}{p^4}}}. \tag{19.6}$$

The magnification factor μ remains finite for all values of the frequency of the disturbance ω; this makes the result obtained radically different from the solution derived above when no inelastic resistance is involved. The relation between μ and the frequency ratio ω/p for various values of n/p is shown in Fig. 121*a*. The general shape of the curves bears a resemblance to Fig. 105*a*; the maximum value of the magnification factor, however, remains finite at $\omega = p$. The

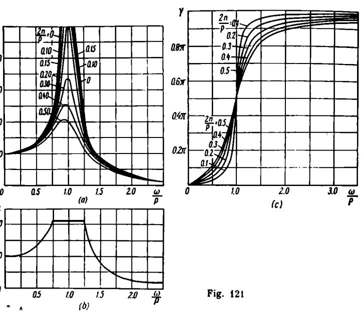


Fig. 121

maximum of the magnification factor is somewhat shifted from the abscissa $\omega/p = 1$. This shift is, however, small and $\mu_{\max}$ can be determined approximately by substituting $\omega = p$ in formula (19.6), giving $\mu_{\max} = p/2n$.

Hence, the maximum value of the magnification factor is inversely proportional to the damping factor n. From the graphs in Fig. 121*a* it is seen that the viscous-damping forces have a pronounced effect only in the nearresonance region. This makes it possible to assume for μ the curve plotted for no viscous resistance (Fig. 105*a*) far from resonance and to assume $\mu=\mu_{\max}$ (Fig. 121*b* for $2n/p=0.5$) in the entire nearresonance region.

Consider the question of vibration "lagging". The phase angle γ defined by formula (19.5) depends on the frequency ratio ω/p, as shown in Fig. 121*c*. As is seen, the angle γ is small at low frequencies ω. At resonance ($\omega = p$) the phase angle is equal to $\pi/2$, i.e., when the force is maximum the displacement is zero. At very high frequencies the phase angle is close to π, i.e., to the maximum force corresponds the maximum displacement.

The viscous resistance affects appreciably the total force transmitted to the foundation during vibration. This force is made up of two components corresponding to the elastic and viscous constraint (Fig. 115*a*)

$$N = kx + c\dot{x}. \tag{19.7}$$

Substituting

$$x = \mu x_{st} \sin(\omega t - \gamma), \quad \dot{x} = \mu\omega x_{st} \cos(\omega t - \gamma).$$

we obtain

$$N = k\mu x_{st}\left[\sin(\omega t - \gamma) + \frac{2n\omega}{p^2}\cos(\omega t - \gamma)\right].$$

The maximum value of N is found to be

$$N_{\max} = \mu P_0 \sqrt{1 + \frac{4n^2\omega^2}{p^4}} = \mu_* P_0,$$

where

$$\mu_* = \mu\sqrt{1 + \frac{4n^2\omega^2}{p^4}}.$$

The quantity μ_* is called the *transmissibility*. Figure 122 shows the variation of this factor with the ratio ω/p for various values of $2n/p$. All curves pass through one and the same points whose abscissa is $\sqrt{2}$ and ordinate 1. In the region $\omega/p < \sqrt{2}$ damping is useful since it reduces the transmissibility; in the region $\omega/p > \sqrt{2}$ the transmissibility increases with increasing amounts of damping. Therefore, when the structure works in the post-resonance region, the force transmitted to the foundation increases due to damping.

The physical meaning of this phenomenon can be understood from Fig. 115*a*. During vibration two forces, as it were, are transmitted to the foundation, through the spring and the dashpot; high frequencies of disturbance involve relatively high velocities and accordingly relatively large viscous-damping forces.

Example 24. From a free-vibration test of an automobile it is found that (1) the natural frequency is 7.33 $\sec^{-1}$, (2) the amplitudes form a geometric progression, the ratio between two consecutive amplitudes being, on the average, 3.1. Determine the maximum magnification factor that can be obtained during forced vibration of this system.

We find the period of free vibration

$$T=\frac{2\pi}{p}=\frac{6.28}{7.33}=0.856 \text{ sec.}$$

Substituting the value of the period T and the amplitude ratio in the expression for the logarithmic decrement (5.6), we obtain

$$n=\frac{\ln 3.1}{0.856}=1.32.$$

The maximum magnification factor [see formula (19.6)] is

$$\mu_{\max}=\frac{p}{2n}=\frac{7.33}{2\times 1.32}=2.77.$$

Note. In solving this problem no account was taken of the difference between the frequency p_* found from the test and the frequency p, which should be substituted in the formula for $\mu_{\max}$. This difference is not very great; indeed, if

$$p_*=\sqrt{p^2-1.32^2}=7.33 \text{ sec}^{-1},$$

then

$$p=7.45 \text{ sec}^{-1}.$$

Effect of energy properties of sources of excitation. In many cases resonance phenomena are detrimental and special precautions should be taken to provide operating conditions of the system which are sufficiently far removed from resonance. But there are manufacturing processes (for example, jigging conveyors) which are based on the utilization of vibrations. For efficiency of such processes, it is desi-

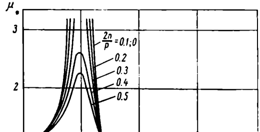


Fig. 122

rable to have the largest possible amplitudes of vibration; this can be achieved by creating resonance or nearresonance conditions.

Experience shows that the operation of such systems in the resonance region is sometimes unstable and the system "spontaneously" leaves this region. These phenomena were explained not so long ago and were shown to be related to the energy properties of sources of excitation.

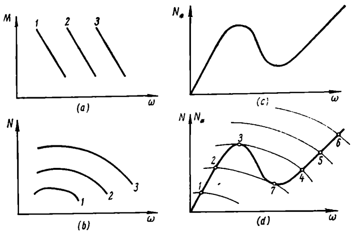


Fig. 123

Figure 123*a* shows a family of characteristics typical of a shunt electric motor. Here, the abscissa represents the speed ω of the armature and the ordinate the torque which can be developed by the motor at a specified speed ω. As the position of the adjuster is varied (by changing the resistance in the exciting circuit), a certain characteristic is fixed (1, 2, 3, . . .).

The characteristics make it possible to determine the power by the formula $N = M\omega$. Figure 123*b* shows the power curves corresponding to the characteristics of Fig. 123*a*. To each position of the adjuster (rheostat) there also corresponds a definite curve (1, 2, 3, . . .).

We now turn our attention to the determination of energy losses. For simplicity, we assume that these losses are due only to the vibrating system itself and the friction is of a viscous type. The energy Ψ dissipated per cycle is given by formula (5.11); substituting $n = 1$, $S = \pi$ and the frequency ω for the natural frequency p, we obtain

$\Psi = \pi c \omega a^2$. Dividing this quantity by the period of forced vibration $2\pi/\omega$, we find the power actually lost in vibration, $N_* = \frac{1}{2} c\omega^2 a^2$. Substituting $c = 2mn$ and the expression for the amplitude a as given by formula (19.4), we have

$$N_* = \frac{mn\omega^2 x_{st}^2}{\left(1-\frac{\omega^2}{p^2}\right)^2 + \frac{4n^2\omega^2}{p^4}} .$$

When the amplitude of the driving force is proportional to ω^2 (see pp. 207-208), x_{st} is also proportional to ω^2 and the curve $N_*(\omega)$ assumes a shape as shown in Fig. 123*c*.

We can now correlate the quantities N and N_*; this is done in Fig. 123*d*. Suppose that the rheostat is set so that curve *1* is the characteristic of the motor; the operating conditions then correspond to point *1*.

Let now the rheostat be switched out slowly and the system gradually accelerated. From Fig. 123*d* it is seen that as the rheostat is gradually switched out the curves $N(\omega)$ rise higher and higher. Accordingly, after conditions *1* and *2* are passed the system is rapidly accelerated at point *3* until condition *4* is reached. Further switching out of the rheostat sets up conditions *5*, *6*, etc.

As the machine is gradually stopped and the resistance in the exciting circuit gradually increases, the vibrating system will pass successively through conditions *6*, *5*, *4*, *7*, *2*, *1*, etc.; the transition from condition *7* to condition *2* will take place as a sudden decrease in the frequency ω.

As can be seen, in the case of either acceleration or stopping-down the branch of the curve *3-7* remains *unrealized*. The conditions on this branch are not feasible even if the power N is reduced after point *3* is reached during acceleration; they are found to be unstable and the system passes from these conditions to stable ones. Thus, the frequency range (ω_3, ω_7) is not practicable at all.

Clearly, this phenomenon occurs in all cases where the curves of available power N may intersect the curve of required power N_* in more than one point. It should be noted that in such cases the point at the top of the resonance peak will always be a point of unstable state; this is the cause of the phenomena discussed above.

Effect of periodic impulses. As a starting expression we assume the law of damped free vibration (5.5*a*) instead of expression (18.33) and form the expression for the velocity

$$v = \dot{x} = e^{-nt}\left(v_0 \cos p_* t - \frac{nv_0 + p^2 x_0}{p_*} \sin p_* t\right) .$$

As before, the origin of time is chosen at time t_0 (see Fig. 117*b*). For time t_1 we can write

$$x_1 = e^{-nT}\left(x_0 \cos p_* T + \frac{v_0 + nx_0}{p_*} \sin p_* T\right),$$

$$v_1 = e^{-nT}\left(v_0 \cos p_* T - \frac{nv_0 + p^2 x_0}{p_*} \sin p_* T\right).$$

At time t_2 the displacement and velocity are again equal to x_0 and v_0

$$\left.\begin{aligned} x_0 &= x_1 = e^{-nT}\left(x_0 \cos p_* T + \frac{v_0 + nx_0}{p_*} \sin p_* T\right), \\ v_0 &= v_1 + \frac{S}{m} = e^{-nT}\left(v_0 \cos p_* T - \frac{nv_0 + p^2 x_0}{p_*} \sin p_* T\right) + \frac{S}{m}, \end{aligned}\right\} \tag{19.8}$$

where S is the magnitude of the impulse.

From Eqs. (19.8) we find

$$\left.\begin{aligned} x_0 &= \frac{S e^{nT} \sin p_* T}{m p_* \left(e^{2nT} - 2e^{nT} \cos p_* T + 1\right)}, \\ v_0 &= \frac{S e^{nT}\left(e^{nT} - \cos p_* T - \frac{n}{p_*} \sin p_* T\right)}{m\left(e^{2nT} - 2e^{nT} \cos p_* T + 1\right)}. \end{aligned}\right\} \tag{19.9}$$

After calculating x_0 and v_0 by these formulas, we can use the solution given in formula (5.5*a*).

Of particular interest are the *resonance conditions* when the period of impulses T is an integer number of the natural periods of vibration T_*; denoting this number by r, we have

$$T = rT_* = \frac{2\pi r}{p_*}.$$

Then

$$\sin p_* T = 0, \quad \cos p_* T = 1$$

and we find by formulas (19.9)

$$x_0 = 0, \quad v_0 = \frac{S e^{nT}}{m\left(e^{nT} - 1\right)}.$$

For small values of nT we can put

$$e^{nT} \simeq nT + 1, \text{ i.e., } v_0 = \frac{S}{mnT}$$

and the solution takes the form

$$x = \frac{S}{2\pi mnr} e^{-nt} \sin p_* t.$$

The maximum value (resonant amplitude) is approximately equal to

$$x_{\max}=\frac{S}{2\pi mnr},$$

i.e., it is inversely proportional to the damping factor (just as in the case of harmonic disturbance). The repetition factor at resonance is obtained by dividing $x_{\max}$ by the amplitude of the vibration produced by a single impact (S/mp)

$$\beta=\frac{p}{2\pi nr},$$

i.e., the resonant amplitudes diminish with increasing r (with decreasing frequency of impulses).

Figure 124 shows a curve similar to that of Fig. 118 but with inelastic resisting forces taken into account (for the case $n/p = 0.1$).

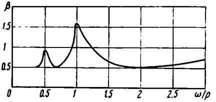


Fig. 124

Effect of an arbitrary periodic force. The computations corresponding to this case are basically the same as those presented on p. 225-226 for a simpler case when there is no damping. The final formula is somewhat more complicated than formula (18.35*a*) but has the same structure

$$x(t)=\frac{e^{-nt}}{mp_*}\times$$
$$\times\left\{\frac{C\,[e^{nt}\sin p_*(t+T)-\sin p_*t]-S\,[e^{nt}\cos p_*(t+T)-\cos p_*t]}{1-2e^{nT}\cos p_*T+e^{2nT}}+\right.$$
$$\left.+\int_0^t P(\tau)\,e^{n\tau}\sin p_*(t-\tau)\,d\tau\right\},$$

where

$$C=\int_0^T P(\tau)\,e^{n\tau}\cos p_*\tau\,d\tau,$$
$$S=\int_0^T P(\tau)\,e^{n\tau}\sin p_*\tau\,d\tau.$$

To determine the displacements it is sufficient to carry out the integration over an interval equal to the period of variation of the driving force.

It should be emphasized that these formulas have definite advantages over the relations which are obtained when the driving force is expanded in a Fourier series since they eliminate the necessity for using infinite (and often slowly convergent) series.

Effect of a force of variable frequency. An example of a harmonic driving force produced by an unbalanced rotor was given above (see Fig. 9); it was then assumed that the speed of rotation was constant. Consider the vibrations induced during acceleration of a machine when the speed gradually increases from zero to some final value. Of particular importance is the case when acceleration involves *passage through resonance.* If the passage is not very slow, the resulting vibrations differ greatly from the vibrations under steady-state conditions. Therefore, it would be wrong to assess the danger of passing through resonance from the amplitudes calculated by a consideration of steady-state resonant vibrations.

If the angular velocity of rotation is assumed to increase uniformly and the angular acceleration is denoted by α, the angular velocity is αt and the angle of rotation is $\frac{1}{2}\alpha t^2$. Accordingly, the centrifugal force is equal to $mr(\alpha t)^2$, where mr is the product of the rotating mass and the eccentricity; the vertical component of this force (driving force) is then

$$P(t) = mr(\alpha t)^2 \sin\frac{\alpha t^2}{2}.$$

The vibrations produced by this force can be described by means of the general solution (19.2). The resulting integral, however, is not expressible in terms of elementary functions; even in an artificially simplified problem when there is no friction and the amplitude of the force is assumed to be constant at all times during acceleration, the solution leads to transcendental functions called *Fresnel integrals.* Without going through rather complicated calculations, we present the final basic conclusions which pertain to the case of a force of constant amplitude acting on a system with viscous friction:

the peak amplitudes of vibration are reached not when the frequency of the driving force coincides with the natural frequency of the system but somewhat later; during acceleration the peak amplitude is shifted toward higher frequencies, and during stopping-down toward lower frequencies;

the peak amplitudes of vibration when passing through resonance are smaller than in the case of steady-state resonant vibrations, the difference being greater the shorter the time of passage.

Figure 125a shows the variation of a force P following the law

$$P = P_0 \sin \frac{\alpha t^2}{2},$$

where the "dimensionless" time $\alpha t/p$ is taken as the argument. Figure 125b shows vibration curves corresponding to various rates of acceleration; the dimensionless parameter α/p^2 is taken as a measure

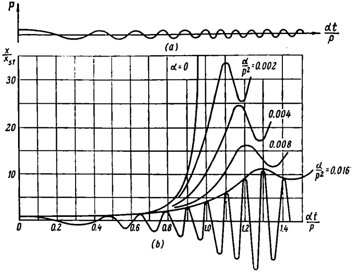


Fig. 125

of the rate of acceleration. The higher the rate of acceleration, the smaller is the peak amplitude and the greater the shift to the right.

Figure 126a shows graphs indicating the frequency $\bar{\omega}$ at which the peak amplitudes are reached in cases of direct and return passage through resonance. The abscissa represents the value of the characteristic parameter $10^3\alpha/p^2$; here the corresponding number of cycles N of the driving force after which this peak amplitude occurs can also be determined (the curves apply to the case $2n/p = 0.05$).

From the graphs of Fig. 126b it is possible to determine the ratio between the maximum magnification factor $\bar{\mu}_{max}$ reached when passing through resonance and the value of μ_{max} which corresponds to the steady-state resonant vibrations.

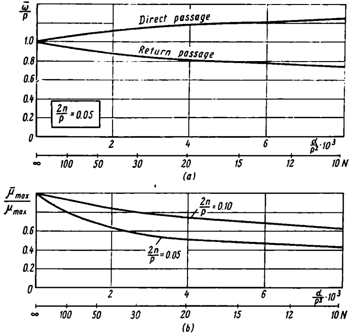


Fig. 126

Effect of Arbitrarily Given Inelastic Resisting Forces

Consider the general case when the inelastic resisting force is a non-linear function of the velocity, $R = R(\dot{x})$.

Because of the complexity involved in a rigorous consideration of the effect of such a force we restrict ourselves to an approximate, but sufficiently accurate, simple procedure. Replace the force R by an *equivalent* viscous-damping force

$$R_* = c_* \dot{x} \qquad (19.10)$$

and determine the coefficient c_* by equating the work done per cycle by the forces R and R_*.

We have to make a certain assumption as to the type of vibratory process. For a harmonic driving force it is natural to assume that in the general case of inelastic resisting forces the vibratory process is also governed by law (19.3).

It will be found more convenient to shift the origin of time so that the law of vibration takes a simpler form

$$x = a \sin \omega t \qquad (19.11)$$

and to require the equality of the work done per half-cycle $\left(0, \frac{T}{2}\right)$ when the velocity (and also the forces R and R_*) maintains a constant sign.

The elementary work of the equivalent force R_* is then given by

$$R_* \, dx = R_* \dot{x} \, dt. \qquad (19.12)$$

Substituting expression (19.10), we obtain

$$c_* \dot{x}^2 \, dt = a^2 \omega^2 c_* \cos^2 \omega t \, dt.$$

Accordingly, the work done by the force R_* per cycle is equal to

$$\int_0^T R_* \dot{x} \, dt = \pi c_* a^2 \omega. \qquad (19.13)$$

The work done by the given non-linear inelastic resisting force must be expressed in a similar way.

Suppose that these operations have been carried out and the equivalent coefficient c_* determined (its magnitude, as a rule, is dependent on the amplitude of vibration a).

Substitute this expression for c_* in solution (19.4)

$$a = \frac{x_{st}}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{c_* \omega}{k}\right)^2}}. \qquad (19.14)$$

It should be noted that the unknown amplitude appears on both sides of this equality. After determining the relation between the amplitude and the frequency of vibration ω from expression (19.14), we can plot a curve similar to those in Fig. 121*a*.

Let us carry through these operations for inelastic resisting forces of the form

$$R = c\dot{x} \, |\dot{x}|^{n-1}. \qquad (19.15)$$

Similarly to expression (19.12), the elementary work of force (19.15) is

$$R \, dx = R\dot{x} \, dt = c\dot{x}^2 \, |\dot{x}|^{n-1} \, dt.$$

Substituting expression (19.11), we obtain

$$c\dot{x}^2\,|\,\dot{x}\,|^{n-1}\,dt = ca^{n+1}\omega^{n+1}\cos^2\omega t\,|\cos\omega t\,|^{n-1}\,dt;$$

the work done by the force R per cycle is equal to

$$\int\limits_0^T R\dot{x}\,dt = ca^{n+1}\omega^{n+1}\int\limits_0^T \cos^2\omega t\,|\cos\omega t\,|^{n-1}\,dt = ca^{n+1}\omega^n\int\limits_0^{2\pi}|\cos\psi\,|^{n+1}\,d\psi.$$

The integral appearing here is also equal to S [see formula (5.12)], so that

$$\int\limits_0^T R\dot{x}\,dt = ca^{n+1}\omega^n S. \tag{19.16}$$

Equating expressions (19.13) and (19.16)

$$\pi c_* a^2\omega = ca^{n+1}\omega^n S,$$

we find the *equivalent damping constant*

$$c_* = \frac{cS}{\pi}(a\omega)^{n-1}$$

which should be substituted in relation (19.14); the latter is used to determine the amplitude of vibration.

Thus, if $n = 2$, from the table on p. 63 we find $S = 2.667$, i.e., $c_* = 0.85ca\omega$. Substituting this expression in formula (19.14), we obtain

$$a = \frac{x_{st}}{\sqrt{\left(1-\frac{\omega^2}{p^2}\right)^2+\left(\frac{0.85ca\omega^2}{k}\right)^2}}.$$

Solving this biquadratic equation, we obtain the final expression for the amplitude of vibration

$$a = \frac{k}{1.20c\omega^2}\sqrt{\sqrt{\left(1-\frac{\omega^2}{p^2}\right)^4+4x_{st}\left(\frac{0.85c\omega^2}{k}\right)^2}-\left(1-\frac{\omega^2}{p^2}\right)^2}.$$

The maximum amplitude corresponds approximately to the equality $\omega = p$ and is given by

$$a_{\max} \cong 1.085\sqrt{\frac{mx_{st}}{c}}.$$

Effect of Hysteresis Losses

Using formula (5.1) and assuming the shape of the hysteresis loop to be elliptic (see Fig. 30), we find that the inelastic resisting forces are defined by the formula

$$R = \pm\frac{ca^n}{\pi}\sqrt{1-\left(\frac{x}{a}\right)^2}.$$

The equation of forced vibrations produced by a harmonic driving force $P_0 \sin \omega t$ is written as

$$\ddot{x} + p^2 x \pm \frac{ca^n}{\pi m} \sqrt{1 - \left(\frac{x}{a}\right)^2} = \frac{P_0}{m} \sin \omega t. \qquad (19.17)$$

This non-linear equation has a very simple exact solution

$$x = a \sin (\omega t - \gamma), \qquad (19.18)$$

where a and γ are suitably chosen quantities. Substitute expression (19.18) in Eq. (19.17)

$$- a\omega^2 \sin (\omega t - \gamma) + ap^2 \sin (\omega t - \gamma) + \\ + \frac{ca^n}{\pi m} \cos (\omega t - \gamma) = \frac{P_0}{m} \sin \omega t. \qquad (19.19)$$

The last term on the left-hand side is written with the plus sign only, since the change of sign required for the equation (when the velocity changes sign) is automatically provided here by the changing sign of the cosine. Indeed, the term $\frac{ca^n}{\pi m} \cos (\omega t - \gamma)$ is in phase with the velocity which, according to expression (19.18), is equal to

$$\dot{x} = a\omega \cos (\omega t - \gamma).$$

Transforming relation (19.19), we obtain

$$\left[a (p^2 - \omega^2) \cos \gamma + \frac{ca^n}{\pi m} \sin \gamma - \frac{P_0}{m}\right] \sin \omega t - \\ - \left[a (p^2 - \omega^2) \sin \gamma - \frac{ca^n}{\pi m} \cos \gamma\right] \cos \omega t = 0.$$

Since this equality must be satisfied identically, the bracketed expressions must be equal to zero, i.e.,

$$a (p^2 - \omega^2) \cos \gamma + \frac{ca^n}{\pi m} \sin \gamma - \frac{P_0}{m} = 0,$$

$$a (p^2 - \omega^2) \sin \gamma - \frac{ca^n}{\pi m} \cos \gamma = 0.$$

Hence

$$a = \frac{x_{st}}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{ca^{n-1}}{\pi k}\right)^2}}, \qquad (19.20)$$

$$\tan \gamma = \frac{ca^{n-1}}{\pi m (p^2 - \omega^2)}. \qquad (19.21)$$

In three particular cases when $n = 0, 1, 2$, the solution can be obtained immediately from Eq. (19.20).

When $n = 2$ relation (19.20) reduces to a biquadratic equation from which we obtain

$$a = \frac{\pi m (p^2 - \omega^2)}{\sqrt{2} c} \sqrt{\sqrt{\frac{4P_0 c^2}{\pi^2 m^4 (p^2 - \omega^2)^4}} + 1 - 1}.$$

When $n = 1$ we find from relation (19.20)

$$a = \frac{x_{st}}{\sqrt{\left(1 - \frac{\omega^2}{p^2}\right)^2 + \left(\frac{c}{\pi k}\right)^2}}.$$

Note the similar structure of the last formula and formula (19.4) derived for the case of viscous resistance.

When $n = 0$ relation (19.20) gives

$$a = \frac{x_{st}}{1 - \frac{\omega^2}{p^2}} \sqrt{1 - \frac{c^2}{\pi^2 P_0^2}}.$$

From this formula it is seen that the amplitude builds up without limit when the frequencies coincide, $\omega = p$.

Away from resonance, the second term under the radical in expression (19.20) is small in comparison with the first term; this makes it possible to construct the process of successive approximations for calculating the amplitude for any value of n. After determining the amplitude a, we can find the phase angle γ from formula (19.21).

For any value of n the resonant amplitude is determined from expression (19.20) as

$$a_{\max} = \sqrt[n]{\frac{\pi P_0}{c}}.$$

20. Systems with Non-Linear Restoring Forces (Single Degree of Freedom)

Harmonic Driving Force

Basic equation. Consider forced vibrations of a system with a non-linear restoring force under the action of a harmonic driving force, assuming that there are no inelastic resistances. The corresponding equation has the standard form

$$\ddot{x} + \frac{F(x)}{m} = \frac{P_0}{m} \sin \omega t, \tag{20.1}$$

where $F(x)$ is a non-linear restoring force (Fig. 127).

Equation (20.1) is not solvable in closed form; the methods of solution given below are approximate.

Graphical solution. The general form of the solution can be determined using a simple, though the least accurate, procedure on the assumption that the vibrations of the system under consideration are governed by the law

$$x = a \sin \omega t, \tag{20.2}$$

as is the case in linear systems. Solution (20.2) is an exact one only when the characteristic is linear, but in the general case the substitution of solution (20.2) in Eq. (20.1) does not reduce it to an identity. Let us require that Eq. (20.1) be fulfilled at least at instants when the force $P(t)$ and the deflection x reach maximum values

$$P_{\max} = P_0, \quad x_{\max} = a.$$

The acceleration is then maximum too

$$\ddot{x}_{\max} = -a\omega^2$$

and, according to Eq. (20.1), we obtain the relation

$$F(a) = P_0 + ma\omega^2. \tag{20.3}$$

The quantity $F(a)$ represents the maximum value of the elastic restoring force (Fig. 128a).

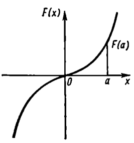


Fig. 127

Relation (20.3) should be regarded as an equation for the unknown amplitude of vibration a. We shall use a graphical method for solving this non-linear equation.

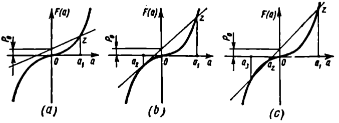


Fig. 128

In the same co-ordinate system in which the curve $F(a)$ is already plotted, we draw a straight line

$$z = P_0 + ma\omega^2 \tag{20.4}$$

using the given values, m, P_0 and ω. Suppose that the frequency ω is not high; then the slope of the straight line will be sufficiently small (Fig. 128a). It is clear that the abscissa of the point of intersection of the curve and the straight line satisfies Eq. (20.3), i.e., it is the only real root of this equation.

The product $m\omega^2$ is the slope of the line. Therefore, if we consider higher values of ω, line (20.4) will be steeper; finally, at a sufficiently high value of ω the z line will touch the curve $F(a)$ in the third quadrant (Fig. 128*b*). At still higher values of ω the straight line will intersect the curve $F(a)$ twice in the third quadrant (Fig. 128*c*). In these cases there are more than one root of the equation: two roots (abscissas a_1 and a_2) for the straight line shown in Fig. 128*b* and three roots (abscissas a_1, a_2 and a_3) for the straight line shown in Fig. 128*c*. The variation of these roots with a gradual increase of the frequency ω is shown in Fig. 129*a*. Of course, the resulting curves *I*, *II* and *III* represent only an approximate solution of the problem but they give a correct qualitative idea of the variation of the amplitude of forced vibration with the frequency ω.

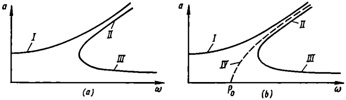


Fig. 129

The following features of the process of forced vibration may be observed in the non-linear system under consideration:

at sufficiently high frequencies ω the solution becomes non-unique; to one and the same value of the frequency ω there may correspond three values of the amplitude;

the amplitude of vibration always remains bounded even when no inelastic resistances are involved.

The relation between the amplitude of free vibration and the natural frequency of the same non-linear system is represented as curve *IV* in Fig. 129*b*.

This curve may be plotted by using a graphical solution similar to that shown in Fig. 128; in this case we have to put $P_0 = 0$ and find the points of intersection of the curve $F(a)$ and the straight line $z = mp^2a$ passing through the origin of co-ordinates (Fig. 130). To each value of the frequency corresponds a single value of the amplitude. Point p_0 (Fig. 129*b*) corresponds to the case of vanishingly small amplitudes of vibration. Curve *IV* indicating the amplitudes of free vibration is a *skeleton line* dividing two branches for the amplitudes of forced vibration.

The curves given in Fig. 129 are typical of systems with a *stiff characteristic* (i.e., of systems with gradually increasing stiffness). If a system has a *soft characteristic* (the stiffness decreases with increasing x), the skeleton line bends to the left; the curves for the amplitudes of forced vibration are distorted accordingly.

Further investigation may show that to the roots a_2 correspond unstable solutions. Hence, the amplitudes a vary with frequency as shown in Fig. 131.

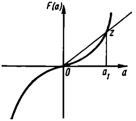


Fig. 130

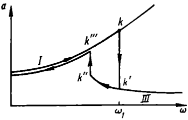


Fig. 131

Initially, as the frequency ω increases gradually, the amplitudes also increase following branch I. If, beginning with some frequency ω_1, the frequency *decreases*, the amplitude suddenly drops to branch III (points k and k') and then the amplitude increases slowly to point k''. With a further decrease in the frequency, at point k'' the amplitude suddenly jumps up (point k''' on branch I) and then decreases gradually, following curve I.

If, however, the frequency ω is *increased* after the sudden drop from branch I to branch III, the variation of the amplitude will follow curve III. Thus, small fluctuations of the frequency about any value ω_1 will invariably bring the generic point to position k'; in this sense branch III corresponds to more stable solutions than branch I.

The foregoing approach clearly reveals the general nature of the process but to obtain reliable quantitative results more accurate procedures are needed which are given below.

Bubnov-Galerkin method. According to this method, we require that an integral similar to (6.21) be zero

$$\int_0^{\frac{2\pi}{\omega}} [m\ddot{x}_0 + F(x_0) - P_0 \sin \omega t]\, x_0\, dt = 0$$

substituting a suitable form of the solution $x_0(t)$. For instance, assuming this solution in the form of expression (20.2), we obtain

$$\int_0^{\frac{2\pi}{\omega}} [-ma\omega^2 \sin\omega t + F(a\sin\omega t) - P_0 \sin\omega t] \sin\omega t\, dt = 0.$$

Carrying out the integration, we obtain a non-linear algebraic equation for the amplitude of vibration a; it can be written in expanded form after specifying the type of characteristic $F(x)$. The general behaviour of the solution corresponds to that established above (see Fig. 129).

Direct linearization method. According to this method, the non-linear equation (20.1) is replaced by a linear one

$$\ddot{x} + p^2 x = \frac{P_0}{m} \sin\omega t \tag{20.5}$$

in which the quantity p^2 is determined as was explained above (see p. 84).

The steady-state part of the solution of the linear equation (20.5) is

$$x = \frac{P_0 \sin\omega t}{m(p^2 - \omega^2)}$$

so that we can write for the amplitude

$$a = \frac{P_0}{m(p^2 - \omega^2)}. \tag{20.6}$$

If, for instance,

$$\frac{F(x)}{m} = p_0^2 x + \alpha x^3,$$

then, as was found above,

$$p^2 = p_0^2 + \frac{5}{7}\alpha a^2$$

and relation (20.6) becomes

$$a = \frac{P_0}{\left(mp_0^2 + \frac{5}{7} m\alpha a^2 - m\omega^2\right)},$$

i.e.,

$$\frac{5}{7} m\alpha a^3 + m(p_0^2 - \omega^2)a = P_0. \tag{20.7}$$

Next, this non-linear equation should be used to determine the unknown amplitude a.

Duffing's method. This method is based on the procedure of eliminating secular terms. Following Duffing, we shall restrict our consideration to a cubic characteristic

$$\frac{F(x)}{m}=p_0^2x+\alpha x^3$$

when the equation governing forced vibrations takes the form

$$\ddot{x}+p_0^2x+\alpha x^3=\frac{P_0}{m}\sin\omega t. \qquad (20.8)$$

Assuming that the solution is a periodic function with period $T=2\pi/\omega$* and denoting by a the unknown magnitude of the maximum deflection, we have the following conditions: $x=a$ and $\dot{x}=0$ at $t=T/4$.

As a first approximation we assume

$$x=a\sin\omega t \qquad (20.9)$$

which is an exact solution in the case of a linear system (i.e., for $\alpha=0$).

Before seeking the next approximation, we rewrite the differential equation (20.8) as

$$\ddot{x}+\omega^2x=(\omega^2-p_0^2)\,x-\alpha x^3+\frac{P_0}{m}\sin\omega t. \qquad (20.10)$$

Substitute solution (20.9) on the right-hand side of Eq. (20.10)

$$\ddot{x}+\omega^2x=\left[(\omega^2-p_0^2)\,a-\frac{3}{4}\,\alpha a^3+\frac{P_0}{m}\right]\sin\omega t+\frac{1}{4}\,\alpha a^3\sin 3\omega t.$$

If the bracketed expression is different from zero, the solution will involve a secular term, giving the resonant condition. To eliminate the secular term it is necessary to put

$$(\omega^2-p_0^2)\,a-\frac{3}{4}\,\alpha a^3+\frac{P_0}{m}=0. \qquad (20.11)$$

This relation, which is identical in form with Eq. (20.7), may be used to determine the amplitude a.

If relation (20.11) is fulfilled, Eq. (20.10) assumes the form

$$\ddot{x}+\omega^2x=\frac{1}{4}\,\alpha a^3\sin 3\omega t.$$

A particular solution of this equation is given by

$$\frac{\alpha a^3\sin 3\omega t}{4\omega^2\left[1-\left(\frac{3\omega}{\omega}\right)^2\right]}=-\frac{\alpha a^3}{32\omega^2}\sin 3\omega t$$

---

* Actually, besides these vibrations, non-linear systems may have *subharmonic vibrations* whose period is an integer number of periods T.

so that the general solution is

$$x = C_1 \sin \omega t + C_2 \cos \omega t - \frac{\alpha a^3}{32\omega^2} \sin 3\omega t.$$

The constants C_1 and C_2 must be determined from the initial conditions given above; using them, we find

$$C_1 = a + \frac{\alpha a^3}{32\omega^2}, \quad C_2 = 0$$

and

$$x = a \sin \omega t + \frac{\alpha a^3}{32\omega^2} (\sin \omega t - \sin 3\omega t).$$

Thus, the harmonic force $P_0 \sin \omega t$ produces in a non-linear system not only vibrations of the fundamental tone with frequency ω but also vibrations with a *higher* frequency.

To make a next approximation it is necessary to substitute expression (20.7) on the right-hand side of Eq. (20.10). Further, to eliminate the possibility of appearing of a secular term the coefficient of $\sin \omega t$ must again be put equal to zero; this gives a refined equation of the type of (20.11) which relates the frequency ω and the amplitude a, etc.

In the development of this method, Duffing (as well as Rayleigh) assumed the amplitude of vibration of the fundamental tone to be constant when passing from one approximation to the next one; therefore he obtained the relation between the frequency ω and the amplitude of the first tone rather than the total amplitude of vibration. The foregoing solution eliminates this defect.

Effect of Viscous Resistance

To assess qualitatively the effect of viscous resistance on forced vibrations of a non-linear system we shall use the approximate procedure which was employed in the case when no inelastic resistances were involved. In the present case it is natural to assume that the vibrations will lag behind the driving force so that if the force varies according to the law $P = P_0 \sin \omega t$, the vibrations are governed by the equation $x = a \sin (\omega t - \alpha)$ to a first approximation. It is possible to describe the vibrations by the equation $x = a \sin \omega t$ assuming the law of variation of the force in the form $P = P_0 \sin (\omega t + \alpha)$; this is done below, the differential equation of vibration being written as

$$m\ddot{x} + c\dot{x} + F(x) = P_0 \sin (\omega t + \alpha) \tag{20.12}$$

and the approximate solution as

$$x = a \sin \omega t. \tag{20.13}$$

As before, we require that Eq. (20.12) be satisfied at the time of maximum deflection ($\omega t = \pi/2$) when $\sin \omega t = 1$. Then

$$x = a, \qquad \dot{x} = 0, \qquad \ddot{x} = -a\omega^2,$$

$$\sin(\omega t + \alpha) = \cos\alpha$$

and the differential equation (20.12) is replaced by the relation

$$-ma\omega^2 + F(a) = P_0 \cos\alpha. \tag{20.14}$$

Also, we require that solution (20.13) satisfy Eq. (20.12) at the time the equilibrium position is passed ($\omega t = 0$) when $\sin(\omega t + \alpha) = \sin\alpha$. Substituting $x = 0$, $F(x) = 0$, $\dot{x} = a\omega$, $\ddot{x} = 0$, $\sin(\omega t + \alpha) = \sin\alpha$ in Eq. (20.12), we obtain

$$ca\omega = P_0 \sin\alpha. \tag{20.15}$$

Thus, Eqs. (20.14) and (20.15) serve to determine the unknowns a and α. Squaring these equations and adding them together, we eliminate the phase angle α and obtain an equation for the unknown amplitude

$$F(a) = \sqrt{P_0^2 - (ca\omega)^2} + ma\omega^2. \tag{20.16}$$

If there are no viscous-damping forces, i.e., $c = 0$, this reduces to the equation (20.3) derived above. For small amplitudes a, the second term under the radical is small compared with the first and hence the solution previously obtained (see Fig. 129) is approximately true in the case under consideration. The difference becomes appreciable for large amplitudes when the second term under the radical sign is close to the first term. This means that the effective value of the force P_0 is small, i.e., the solution approaches that of a free-vibration problem.

Finally, at $ca\omega = P_0$ the radical becomes zero and no further increase of the amplitude is possible. Thus, the largest possible value of the amplitude is

$$a = \frac{P_0}{c\omega}.$$

Equation (20.16) then becomes

$$F(a) = ma\omega^2, \tag{20.17}$$

i.e., it defines the skeleton curve; the maximum amplitude as given by Eq. (20.17) should be marked on this curve.

Figure 132*a* shows an amplitudes curve when damping forces are considered (solid line) and an amplitudes curve without damping forces (dashed line). A gradual increase of the frequency ω leads to the variation of the amplitude in accordance with Fig. 132*b* (solid line); a subsequent decrease of the frequency is shown by a dashed line. In this case there is always a discontinuous jump in the amplitude (point k) even when the frequency ω increases continuously (if there is no damping this jump happens only with decreasing frequency ω).

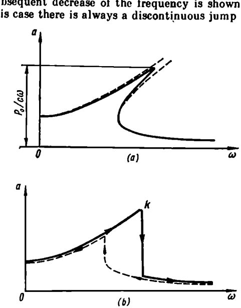


Fig. 132

21. Linear Systems of Several Degrees of Freedom

Principles of Solution

General considerations. If external forces vary according to a periodic law, it is common practice to expand them in a trigonometric series, i.e., represent them as a sum of harmonics; from the principle of superposition, the resultant motion is then determined as the sum of the motions due to each of the harmonics acting separately. With this approach, the solution reduces to the problem involving forced vibrations of the system produced by a single harmonic of the disturbance $P_i \sin \omega t$ (or $P_i \cos \omega t$). Here P_i is the amplitude of the driving force acting in the ith direction; ω is the frequency of the disturbance common to all forces applied to different points of the system.

The solution of this basic problem is usually carried out by one of the two methods: *direct solution* or *expansion in terms of natural modes of vibration*. The method most extensively used in design practice is the second one. An alternate method of solution is by

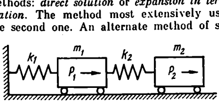


Fig. 133

expansion in terms of natural modes of vibration preserving the given form of periodic loads (i.e., without resolving them into harmonic components).

The features of each of these methods will be illustrated by a simple example of a two-mass system (Fig. 133).

Direct solution. Assuming that the external load is expanded in a trigonometric series, we analyse the motion of the system produced by a single harmonic of the disturbance. The forces acting on each mass are denoted by $P_1 \sin \omega t$ and $P_2 \sin \omega t$. The case when the two forces have the same frequency but different phases $[P_1 \sin(\omega t + \alpha_1)$, $P_2 \sin(\omega t + \alpha_2)]$ will be considered later.

Writing the equations of motion of each of the masses, we obtain, similarly to expressions (7.3) and (7.4),

$$\left.\begin{aligned} P_1 \sin \omega t - k_1 x_1 + k_2 (x_2 - x_1) &= m_1 \ddot{x}_1, \\ P_2 \sin \omega t - k_2 (x_2 - x_1) &= m_2 \ddot{x}_2. \end{aligned}\right\} \qquad (21.1)$$

It is customary to write these equations so that the right-hand sides contain only driving forces

$$\left.\begin{aligned} m_1 \ddot{x}_1 + k_1 x_1 - k_2 (x_2 - x_1) &= P_1 \sin \omega t, \\ m_2 \ddot{x}_2 + k_2 (x_2 - x_1) &= P_2 \sin \omega t. \end{aligned}\right\} \qquad (21.2)$$

The solution of this system, just as the solution of a single equation, is made up of two parts: the solution of the corresponding homogeneous system represented by an expression of the form of (7.14) and a particular solution of the non-homogeneous system (21.2).

As previously noted, the term describing the vibration with the natural frequency decreases rapidly in time due to the action of unavoidable damping forces. Of particular interest is the second part of the solution corresponding to undamped ***steady-state*** forced vibrations.

Assuming a particular solution in the form

$$x_1 = a_1 \sin \omega t, \quad x_2 = a_2 \sin \omega t \tag{21.3}$$

and substituting it in expressions (21.2), we obtain two equations with two unknown amplitudes of vibration a_1 and a_2

$$\left.\begin{aligned} -m_1\omega^2 a_1 + k_1 a_1 - k_2(a_2 - a_1) = P_1, \\ -m_2\omega^2 a_2 + k_2(a_2 - a_1) = P_2. \end{aligned}\right\} \tag{21.4}$$

Solution (21.3) indicates that the vibration occurs with the same frequency as the variation of the forces themselves.

Solving the system of equation (21.4), we find

$$\left.\begin{aligned} a_1 = \frac{P_1(k_2 - m_2\omega^2) + P_2 k_2}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}, \\ a_2 = \frac{P_2(k_1 + k_2 - m_1\omega^2) + P_1 k_2}{(k_1 + k_2 - m_2\omega^2)(k_2 - m_2\omega^2) - k_2^2}. \end{aligned}\right\} \tag{21.5}$$

The denominators of the above expressions are identical with the left-hand side of the frequency equation (7.11*a*) if p is replaced by ω. Consequently, if the frequency of the disturbance ω coincides with either of the two natural frequencies, p_1 or p_2, the denominators in formulas (21.5), according to Eq. (7.11*a*), become zero and the amplitudes a_1 and a_2 become infinite (resonance).

When $\omega = 0$, formulas (21.5) determine the static deflections of the masses produced by the forces P_1 and P_2

$$a_1 = \frac{P_1 + P_2}{k_1},$$

$$a_2 = \frac{P_1 + P_2}{k_1} + \frac{P_2}{k_2}.$$

As $\omega \to \infty$, solutions (21.5) approach zero. The relation between the amplitude a_1 and the frequency is shown in Fig. 134; the graph is plotted for the case $P_1 = 1$, $P_2 = 0$, $k_1 = k_2 = 1$, $m_1 = m_2 = 1$. In this case the number of resonances is two, which corresponds to the number of degrees of freedom of the system and the number of its natural frequencies.

When $\omega = 1$, the amplitude $a_1 = 0$; this case of *antiresonance* is discussed on p. 265.

Using expressions (21.5) it is possible to find the mode of forced vibration defined by the ratio a_2/a_1. In the general case this mode is distinct from either of the natural modes; only at resonance does the mode of forced vibration coincide with the mode of free vibration.

Consider the action of two forces with the same frequency but different phase angles, $P_1 \sin(\omega t + \alpha_1)$ and $P_2 \sin(\omega t + \alpha_2)$.

We can represent these forces as $P_1 \cos \alpha_1 \sin \omega t + P_1 \sin \alpha_1 \cos \omega t$, $P_2 \cos \alpha_2 \sin \omega t + P_2 \sin \alpha_2 \cos \omega t$ and then solve two problems:

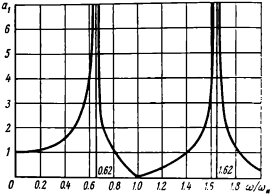


Fig. 134

(1) only the "sine" components $P_1 \cos \alpha_1 \sin \omega t$ and $P_2 \cos \alpha_2 \sin \omega t$ act; (2) only the "cosine" components $P_1 \sin \alpha_1 \cos \omega t$ and $P_2 \sin \alpha_2 \cos \omega t$ act.

In the first problem we obtain the equations

$$-m_1\omega^2 a_1 + k_1 a_1 - k_2 (a_2 - a_1) = P_1 \cos \alpha_1,$$
$$-m_2\omega^2 a_2 + k_2 (a_2 - a_1) = P_2 \cos \alpha_2,$$

and in the second problem (when the particular solution is of the form $x_1 = a_1 \cos \omega t$, $x_2 = a_2 \cos \omega t$)

$$-m_1\omega^2 a_1 + k_1 a_1 - k_2 (a_2 - a_1) = P_1 \sin \alpha_1$$
$$-m_2\omega^2 a_2 + k_2 (a_2 - a_1) = P_2 \sin \alpha_2$$

Each of these problems can be solved in closed form; the results obtained must then be added together.

If the driving forces have a polyharmonic structure

$$P_1 = P_{11} \sin \omega_1 t + P_{12} \sin \omega_2 t + \ldots + P_{1n} \sin \omega_n t,$$
$$P_2 = P_{21} \sin \omega_1 t + P_{22} \sin \omega_2 t + \ldots + P_{2n} \sin \omega_n t,$$

resonance occurs in the following cases:

$$\omega_1 = p_1, \quad \omega_2 = p_1, \quad \omega_3 = p_1, \ \ldots, \ \omega_n = p_1,$$
$$\omega_1 = p_2, \quad \omega_2 = p_2, \quad \omega_3 = p_2, \ \ldots, \ \omega_n = p_2$$

i.e., when any of the n frequencies of the driving force coincides with either of the two natural frequencies of the system.

Solution in terms of natural modes of vibration. Revert to the simplest system subjected to driving forces $P_1 \sin \omega t$ and $P_2 \sin \omega t$.

To derive the basic relationships of the method under consideration, we first form auxiliary relations using Eqs. (7.3) and (7.4). These equations are satisfied both by the solutions

$$\left.\begin{aligned} x_1 &= a_{11} \sin(p_1 t + \alpha_1), \\ x_2 &= a_{21} \sin(p_1 t + \alpha_1) \end{aligned}\right\} \qquad (21.6)$$

and the solutions

$$\left.\begin{aligned} x_1 &= a_{12} \sin(p_2 t + \alpha_2), \\ x_2 &= a_{22} \sin(p_2 t + \alpha_2). \end{aligned}\right\} \qquad (21.7)$$

First substitute solutions (21.6) in Eqs. (7.3) and (7.4)

$$\left.\begin{aligned} -k_1 a_{11} + k_2(a_{21} - a_{11}) &= -m_1 p_1^2 a_{11}, \\ -k_2(a_{21} - a_{11}) &= -m_2 p_1^2 a_{21} \end{aligned}\right\} \qquad (21.8)$$

and then solutions (21.7)

$$\left.\begin{aligned} -k_1 a_{12} + k_2(a_{22} - a_{12}) &= -m_1 p_2^2 a_{12}, \\ -k_2(a_{22} - a_{12}) &= -m_2 p_2^2 a_{22}. \end{aligned}\right\} \qquad (21.9)$$

These relations will be needed in what follows.

In the differential equations (21.2) the unknowns are functions x_1 and x_2. The basic idea of the method under consideration is to replace these functions by two new functions $f_1(t)$, $f_2(t)$ according to the equalities

$$\left.\begin{aligned} x_1 &= a_{11} f_1 + a_{12} f_2, \\ x_2 &= a_{21} f_1 + a_{22} f_2. \end{aligned}\right\} \qquad (21.10)$$

Here a_{11} and a_{12} are arbitrary numbers (we may take, for example, $a_{11} = a_{12} = 1$) to which a_{21} and a_{22} are related by the known expressions (natural modes of vibration). Substituting expressions (21.10) in Eqs. (21.2), we obtain a system of equations in the new functions f_1 and f_2

$$\left.\begin{aligned} & m_1(a_{11}\ddot{f}_1 + a_{12}\ddot{f}_2) + f_1[k_1 a_{11} - k_2(a_{21} - a_{11})] + \\ & \qquad + f_2[k_1 a_{12} - k_2(a_{22} - a_{12})] = P_1 \sin \omega t, \\ & m_2(a_{21}\ddot{f}_1 + a_{22}\ddot{f}_2) + f_1 k_2(a_{21} - a_{11}) + \\ & \qquad + f_2 k_2(a_{22} - a_{12}) = P_2 \sin \omega t. \end{aligned}\right\} \qquad (21.11)$$

This system of equations does not seem at first glance any simpler than the system of equations (21.2). However, the subsequent computations will show that Eqs. (21.11) can be considerably simplified.

To begin with, we rewrite Eqs. (21.11) using relations (21.8) and (21.9)

$$\begin{gathered}m_1(a_{11}\ddot{f}_1+a_{12}\ddot{f}_2)+m_1(p_1^2a_{11}f_1+p_2^2a_{12}f_2)=P_1\sin\omega t,\\ m_2(a_{21}\ddot{f}_1+a_{22}\ddot{f}_2)+m_2(p_1^2a_{21}f_1+p_2^2a_{22}f_2)=P_2\sin\omega t.\end{gathered}\qquad(21.12)$$

Further simplifications follow from the orthogonality properties of the natural modes of vibration. Multiply the first equation of (21.12) by a_{11} and the second by a_{21}; adding the equations obtained together, we find

$$(m_1a_{11}^2+m_2a_{21}^2)\ddot{f}_1+(m_1a_{11}a_{12}+m_2a_{21}a_{22})\ddot{f}_2+p_1^2f_1(m_1a_{11}^2+m_2a_{21}^2+ \\ +p_2^2f_2(m_1a_{11}a_{12}+m_2a_{21}a_{22})=(P_1a_{11}+P_2a_{21})\sin\omega t.$$

According to the orthogonality condition

$$m_1a_{11}a_{12}+m_2a_{21}a_{22}=0$$

and the function f_2 together with its second derivative $\ddot{f}_2$ falls out of the above equation; as a result, we obtain a differential equation in the function f_1 alone

$$\ddot{f}_1+p_1^2f_1=\frac{P_1a_{11}+P_2a_{21}}{m_1a_{11}^2+m_2a_{21}^2}\sin\omega t.\qquad(21.13)$$

In exactly the same way we can obtain a differential equation in the function f_2; to do this, it is necessary to multiply the first equation of (21.12) by a_{12}, the second by a_{22} and add the equations obtained together. Using then the same orthogonality property, we obtain

$$\ddot{f}_2+p_2^2f_2=\frac{P_1a_{12}+P_2a_{22}}{m_1a_{12}^2+m_2a_{22}^2}\sin\omega t.\qquad(21.14)$$

Thus, the method of solution in terms of natural modes of vibration leads to the *separate equations* (21.13) and (21.14) each of which describes the vibration of a single-degree-of-freedom system.

If the right-hand sides of the differential equations (21.13) and (21.14) are denoted by $F_1\sin\omega t$ and $F_2\sin\omega t$, respectively, where

$$F_1=\frac{P_1a_{11}+P_2a_{21}}{m_1a_{11}^2+m_2a_{21}^2},$$
$$F_2=\frac{P_1a_{12}+P_2a_{22}}{m_1a_{12}^2+m_2a_{22}^2},$$

the steady-state part of the solution is

$$f_1=\frac{F_1}{p_1^2-\omega^2}\sin\omega t,$$
$$f_2=\frac{F_2}{p_2^2-\omega^2}\sin\omega t.$$

Now, using relations (21.10) we form [the solutions for the co-ordinates x_1 and x_2

$$x_1 = \left(\frac{F_1 a_{11}}{p_1^2 - \omega^2} + \frac{F_2 a_{12}}{p_2^2 - \omega^2} \right) \sin \omega t,$$
$$x_2 = \left(\frac{F_1 a_{21}}{p_1^2 - \omega^2} + \frac{F_2 a_{22}}{p_2^2 - \omega^2} \right) \sin \omega t.$$

Note that the foregoing procedures ensure the separation of equations also in the case of a larger number of degrees of freedom.

Solution in terms of natural modes of vibration preserving the given form of periodic loads. The main advantage of the above method, viz. separation of equations, is in no way related to a specific form of driving forces. In other words, the separation of equations is as easily achieved in the case of arbitrarily given driving forces $P_1(t)$, $P_2(t)$ as in the above case of harmonic driving forces $P_1 \sin \omega t$ and $P_2 \sin \omega t$. Without repeating the computations, we present the final equations for the general case

$$\ddot{f}_1 + p_1^2 f_1 = \frac{P_1(t)\, a_{11} + P_2(t)\, a_{21}}{m_1 a_{11}^2 + m_2 a_{21}^2},$$
$$\ddot{f}_2 + p_2^2 f_2 = \frac{P_1(t)\, a_{12} + P_2(t)\, a_{22}}{m_1 a_{12}^2 + m_2 a_{22}^2}.$$

In Sec. 18 we saw that such equations could be integrated without much difficulty for any form of right-hand sides; the case of an arbitrary periodic driving force was considered in the same section. Thus, the method of solution in terms of natural modes of vibration requires no preliminary resolution of driving forces into harmonic components. The determination of harmonic components is a laborious operation which sometimes involves the consideration of a large number of harmonics; this operation is justified when the solution is carried out by the first method but seems to be optional or even unnecessary if expansion in terms of natural modes of vibration is used.

Torsional Vibrations of Shafts

General. Forced vibrations of shafts are an inevitable consequence of the variation of torques acting on the shafts; these torques are of a periodic type and are due both to the gas pressure in cylinders and the inertia forces of moving parts.

Consider the most general case when the individual disks of an equivalent system (see Fig. 46*a*) are subjected to arbitrarily given variable torques. The mean values of these torques produce a constant deformation of the shaft; therefore, to analyse vibrations it is sufficient to take into account the effect of only the variable portion of each torque. These portions are denoted by $M_1(t)$, $M_2(t)$, ..., $M_n(t)$ and referred to as *driving torques*.

If no inelastic resistances are assumed, the equations of forced vibrations will differ from the equations of free vibrations (7.18) by the presence of driving torques

$$\left.\begin{aligned} M_1(t)+k_1(\varphi_2-\varphi_1)&=J_1\ddot{\varphi}_1,\\ M_2(t)+k_2(\varphi_3-\varphi_2)-k_1(\varphi_2-\varphi_1)&=J_2\ddot{\varphi}_2,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\\ M_n(t)-k_{n-1}(\varphi_n-\varphi_{n-1})&=J_n\ddot{\varphi}_n. \end{aligned}\right\} \qquad (21.15)$$

Any of the three methods presented above can be used for the solution of this system of equations.

Direct solution. To apply this method it is necessary first to expand the periodic driving torques in Fourier series. Then Eqs. (21.15) are solved several times, for each harmonic individually. This leads to a number of particular problems of the same type, each of which involves the analysis of the action of driving torques of the same frequency $s\omega$

$$\left.\begin{aligned} M_{1s}(t)&=M_1\sin s\omega t,\\ M_{2s}(t)&=M_2\sin s\omega t,\\ \cdots\cdots&\cdots\cdots\\ M_{ns}(t)&=M_n\sin s\omega t. \end{aligned}\right\} \qquad (21.16)$$

For this type of disturbance the steady-state vibrations will have the frequency of the disturbance

$$\left.\begin{aligned} \varphi_1&=a_1\sin s\omega t,\\ \varphi_2&=a_2\sin s\omega t,\\ \cdots&\cdots\cdots\\ \varphi_n&=a_n\sin s\omega t. \end{aligned}\right\} \qquad (21.17)$$

Substituting expressions (21.16) and (21.17) in Eqs. (21.15), we obtain a system of algebraic equations

$$\left.\begin{aligned} -J_1s^2\omega^2a_1-k_1(a_2-a_1)&=M_{1s},\\ -J_2s^2\omega^2a_2+k_1(a_2-a_1)-k_2(a_3-a_2)&=M_{2s},\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\\ -J_ns^2\omega^2a_n+k_{n-1}(a_n-a_{n-1})&=M_{ns}. \end{aligned}\right\} \qquad (21.18)$$

Solving this system, we find the amplitudes $a_1, a_2, \ldots, a_n$ and then the twisting moments at the sections of the shaft: $k_1(a_2-a_1)$ in the first portion, $k_2(a_3-a_2)$ in the second portion, etc.

In solving the system of algebraic equations (21.18), it is advisable to take advantage of the special features of its *chain* structure since each equation involves no more than three unknowns. One can find in the literature a variety of recommendations on the simplification of computations (methods of dynamic stiffnesses, continued fractions, etc).

In particular, the following modification of the Holzer-Tolle method is possible. Adding the first i equations of (21.18) together, we obtain, similarly to system (7.26),

$$k_i(a_{i+1}-a_i) = -s^2\omega^2 \sum_{k=1}^{i} J_k a_k - \sum_{k=1}^{i} M_k. \qquad (21.19)$$

Each of these relations expresses that the twisting moment at a section of the ith portion of the shaft (left-hand side) is equal to the sum of the moments of inertia forces and the external moments for all disks located to the left (right-hand side); of course, here are meant the amplitude values of all the above quantities. From expression (21.19) we obtain a formula similar to expression (7.26*a*)

$$a_{i+1} = -\frac{1}{k_i}\left(s^2\omega^2 \sum_{k=1}^{i} J_k a_k + \sum_{k=1}^{i} M_k\right) + a_i. \qquad (21.20)$$

When using this method in the free-vibration problem it was noted that the amplitude of vibration of the first disk could be assigned arbitrarily and the frequency was the unknown quantity. In the forced-vibration problem here considered the frequency ω should be assumed known and the first amplitude a_1 should be taken as the basic unknown quantity. Assuming successively $i = 1$, $i = 2, \ldots$ in formula (21.20), we can express first a_2 in terms of a_1, then a_3 in terms of a_1, etc. The last formula of the system of equations (21.18) is used to determine the unknown a_1, whereupon the other amplitudes are easily calculated.

These computations are usually performed using a table similar in form to that on p. 103.

Solution in terms of natural modes of vibration. This method requires the frequencies and modes of free vibration to be first calculated, after which the determination of forced vibrations becomes relatively simple. The justification of the computing procedure may seem somewhat cumbersome but is actually very simple and identical with that given above. We shall assume as known quantities both the natural frequencies $p_1, p_2, \ldots, p_{n-1}$ and the corresponding natural modes of vibration (i.e., the ratios between the amplitudes $a_{11}, a_{21}, \ldots, a_{n1}$ of the first natural mode, $a_{12}, a_{22}, \ldots, a_{n2}$ of the second natural mode, etc.).

The system of functions defining the ith principal vibration

$$\left.\begin{aligned} \varphi_1 &= a_{1i}\sin(p_i t + \alpha_i), \\ \varphi_2 &= a_{2i}\sin(p_i t + \alpha_i), \\ &\ldots\ldots\ldots\ldots \\ \varphi_n &= a_{ni}\sin(p_i t + \alpha_i) \end{aligned}\right\} \qquad (21.21)$$

satisfies the system of equations of free vibrations (7.18). Substituting expressions (21.21) in the system of equations (7.18), we obtain the auxiliary relations

$$\left.\begin{aligned} k_1(a_{2i}-a_{1i}) &= -J_1p_i^2a_{1i}, \\ k_2(a_{3i}+a_{2i})-k_1(a_{2i}-a_{1i}) &= -J_2p_i^2a_{2i}, \\ k_3(a_{4i}-a_{3i})-k_2(a_{3i}-a_{2i}) &= -J_3p_i^2a_{3i}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ k_{n-1}(a_{ni}-a_{n-1,\,i})-k_{n-2}(a_{n-1,\,i}-a_{n-2,\,i}) &= -J_{n-1}p_i^2a_{n-1,\,i}, \\ -k_{n-1}(a_{ni}-a_{n-1,\,i}) &= -J_np_i^2a_{ni}. \end{aligned}\right\} \qquad (21.22)$$

Let us make a change of variables in the system of equations (21.22) introducing new functions $f_0(t)$, $f_1(t)$, $\ldots$, $f_{n-1}(t)$ related to the functions $\varphi_1(t)$, $\varphi_2(t)$, $\ldots$, $\varphi_n(t)$ as follows

$$\left.\begin{aligned} \varphi_1 &= f_0+a_{11}f_1+a_{12}f_2+\ldots+a_{1,\,n-1}f_{n-1}, \\ \varphi_2 &= f_0+a_{21}f_1+a_{22}f_2+\ldots+a_{2,\,n-1}f_{n-1}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ \varphi_n &= f_0+a_{n1}f_1+a_{n2}f_2+\ldots+a_{n,\,n-1}f_{n-1}. \end{aligned}\right\} \qquad (21.23)$$

The function $f_0(t)$ common to all the equations of system (21.23) corresponds to the rotation of the system as a *rigid body*, i.e., to the zero natural mode of vibration. The coefficients of the first line a_{11}, $a_{12} = \ldots a_{1,n-1}$ are arbitrarily assumed (for example, $a_{11} = a_{12} = \ldots = 1$); the other coefficients are then determined by the corresponding natural modes.

Substituting expressions (21.23) in the system of equations (21.15), we obtain

$$\begin{aligned} M_1+k_1[(a_{21}-a_{11})f_1+(a_{22}-a_{12})f_2+\ldots+(a_{2,\,n-1}-a_{1,\,n-1})f_{n-1}] &= \\ = J_1(\ddot{f}_0+a_{11}\ddot{f}_1+a_{12}\ddot{f}_2+\ldots+a_{1,\,n-1}\ddot{f}_{n-1}), & \\ M_2+k_2[(a_{31}-a_{21})f_1+(a_{32}-a_{22})f_2+\ldots+(a_{3,\,n-1}-a_{2,\,n-1})]- & \\ -k_1[(a_{21}-a_{11})f_1+(a_{22}-a_{12})f_2+\ldots+(a_{2,\,n-1}-a_{1,\,n-1})f_{n-1}] &= \\ = J_2(\ddot{f}_0+a_{21}\ddot{f}_1+a_{22}\ddot{f}_2+\ldots+a_{2,\,n-1}\ddot{f}_{n-1}), & \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ M_n-k_{n-1}[(a_{n-1}-a_{n-1,\,n})f_1+(a_{n2}-a_{n-1,\,2})f_2+\ldots & \\ \ldots+(a_{n,\,n-1}-a_{n-1,\,n-1})f_{n-1}] &= \\ = J_n(\ddot{f}_0+a_{n1}\ddot{f}_1+a_{n2}\ddot{f}_2+\ldots+a_{n,\,n-1}\ddot{f}_{n-1}). & \end{aligned}$$

These equations can be written in a simpler form if relations (21.22) are taken into account

$$\begin{aligned}
M_1 - J_1(p_1^2 a_{11} f_1 + p_2^2 a_{12} f_2 + \dots + p_{n-1}^2 a_{1,n-1} f_{n-1}) &= \\
= J_1(\ddot{f}_0 + a_{11}\ddot{f}_1 + a_{12}\ddot{f}_2 + \dots + a_{1,n-1}\ddot{f}_{n-1}), \\
M_2 - J_2(p_1^2 a_{21} f_1 + p_2^2 a_{22} f_2 + \dots + p_{n-1}^2 a_{2,n-1} f_{n-1}) &= \\
= J_2(\ddot{f}_0 + a_{21}\ddot{f}_1 + a_{22}\ddot{f}_2 + \dots + a_{2,n-1}\ddot{f}_{n-1}), \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
M_n - J_n(p_1^2 a_{n1} f_1 + p_2^2 a_{n2} f_2 + \dots + p_{n-1}^2 a_{n,n-1} f_{n-1}) &= \\
= J_n(\ddot{f}_0 + a_{n1}\ddot{f}_1 + a_{n2}\ddot{f}_2 + \dots + a_{n,n-1}\ddot{f}_{n-1})
\end{aligned}$$

or more compactly

$$\left.\begin{aligned}
M_1 &= J_1[\ddot{f}_0 + a_{11}(\ddot{f}_1 + p_1^2 f_1) + a_{12}(\ddot{f}_2 + p_2^2 f_2) + \dots \\
&\quad \dots + a_{1,n-1}(\ddot{f}_{n-1} + p_{n-1}^2 f_{n-1})], \\
M_2 &= J_2[\ddot{f}_0 + a_{21}(\ddot{f}_1 + p_1^2 f_1) + a_{22}(\ddot{f}_2 + p_2^2 f_2) + \dots \\
&\quad \dots + a_{2,n-1}(\ddot{f}_{n-1} + p_{n-1}^2 f_{n-1})], \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
M_n &= J_n[\ddot{f}_0 + a_{n1}(\ddot{f}_1 + p_1^2 f_1) + a_{n2}(\ddot{f}_2 + p_2^2 f_2) + \dots \\
&\quad \dots + a_{n,n-1}(\ddot{f}_{n-1} + p_{n-1}^2 f_{n-1})].
\end{aligned}\right\} \quad (21.24)$$

This system of equations falls into independent equations if use is made of the orthogonality property of normal modes. Add all of the equations (21.24) together. The first terms on the right-hand sides give the sum $(J_1 + J_2 + \dots + J_n)\ddot{f}_0$; the second terms add up to the expression $(a_{11}J_1 + a_{21}J_2 + \dots + a_{n1}J_n)(\ddot{f}_1 + p_1^2 f_1)$, which is zero since the sum in the first parentheses vanishes due to the orthogonality of the first normal and the zero mode of vibration. Likewise, the results of summation of all the other terms on the right-hand sides of Eqs. (21.24) become zero. After adding all of the equations (21.24) together we obtain therefore

$$M_1 + M_2 + \dots + M_n = (J_1 + J_2 + \dots + J_n)\ddot{f}_0,$$

whence we can find the function f_0 by integration.

Further, multiplying the first equation of (21.24) by a_{11}, the second by a_{21}, the third by a_{31}, etc., we add all the equations obtained together.

The *first* terms on the right-hand sides form the product

$$(a_{11}J_1 + a_{21}J_2 + \dots + a_{n1}J_n) f_0$$

and the sum in the parentheses is zero. The summation of the *second* terms yields an expression different from zero

$$(a_{11}^2J_1 + a_{21}^2J_2 + \ldots + a_{n1}^2J_n)(\ddot{f}_1 + p_1^2f_1).$$

The *third* terms add up to the expression

$$(a_{11}a_{12}J_1 + a_{21}a_{22}J_2 + \ldots + a_{n1}a_{n2}J_n)(\ddot{f}_2 + p_2^2f_2).$$

Due to the orthogonality of the first two modes of vibration the sum in the first parentheses is zero and so is the whole expression. The sum of the *fourth* terms is also zero, etc.

After these operations the left-hand sides of Eqs. (21.24) give

$$M_1a_{11} + M_2a_{21} + \ldots + M_na_{n1}.$$

We finally obtain a differential equation containing only *one function* f_1

$$\ddot{f}_1 + p_1^2f_1 = \frac{M_1a_{11} + M_2a_{21} + \ldots + M_na_{n1}}{a_{11}^2J_1 + a_{21}^2J_2 + \ldots + a_{n1}^2J_n}.$$

In a similar way it is possible to obtain an equation for the function f_2. To do this, it is necessary to multiply the first equation of (21.24) by a_{12}, the second by a_{22}, the third by a_{32}, etc. and then add all the equations together. We thus obtain the differential equation

$$\ddot{f}_2 + p_2^2f_2 = \frac{M_1a_{12} + M_2a_{22} + \ldots + M_na_{n2}}{a_{12}^2J_1 + a_{22}^2J_2 + \ldots + a_{n2}^2J_n}.$$

Applying this procedure repeatedly, we can form separate equations for the rest of the unknown functions. For the ith function f_i we obtain

$$\ddot{f}_i + p_i^2f_i = \frac{M_1a_{1i} + M_2a_{2i} + \ldots + M_na_{ni}}{a_{1i}^2J_1 + a_{2i}^2J_2 + \ldots + a_{ni}^2J_n}. \qquad (21.25)$$

Equations of this type are most convenient; by this means the vibration problem for a system of n degrees of freedom is replaced by n simple vibration problems for a system of one degree of freedom. Each of these problems is solved by the methods considered in Sec. 18.

In the analysis of torsional vibrations of shafts, the solutions of practical importance are those corresponding to the first two or three natural modes of vibration; this means that it is sufficient to solve two or three equations of the type of (21.25) when $i = 1, 2, 3$.

If the external driving moments are periodic, the right-hand sides of the differential equations are also periodic functions. For further solution it is usual to expand each of the driving torques in a Fourier series, then the effect of each harmonic is analysed and all the results obtained are added together.

In principle all these computations are relatively simple but it should be remembered that they must be repeated for all important harmonic components of the disturbance; the calculation is usually complicated by the fact that the number of harmonics is rather large. The following example gives an idea of the relative importance of different harmonics of the disturbance in the particular case of a four-stroke internal-combustion engine

Number of component . .	1	2	3	4	5	6	7	8	9	10
Amplitude of component	2.38	2.69	2.65	2.31	1.95	1.64	1.01	0.76	0.59	0.47

As is seen, the amplitudes of harmonics decrease very slowly and in this case it is necessary to use no less than 13 to 15 harmonics in the analysis. We emphasize once again that the expansion of driving torques in a Fourier series is unnecessary if the solution is sought by means of Eq. (21.25).

Since the speed of the shaft may vary in service, the frequencies of the disturbance $\omega_s = s\omega$ are not constant; as the rotating conditions vary, so do the frequencies of the disturbance. The coincidence of the frequency of any one of the harmonics of the disturbance with one of the natural frequencies becomes a real possibility. In case of such coincidence the system is in resonance and *inelastic resisting forces* should be introduced in the analysis of the amplitudes of vibration.

The complete solution of the problem, even on the simplest assumption of viscous friction forces, is found to be very cumbersome. In practice, therefore, the calculations are performed by approximate methods. The principal simplification is that the mode of vibration at resonance is usually assumed to be coincident with the corresponding natural mode determined without damping forces.

Let, for example, one of the harmonics of the driving force have a frequency ω_s equal to the ith natural frequency; in the analysis of vibrations only the ith natural mode is then taken into account and if the damping is of a viscous type, we obtain in place of Eq. (21.25)

$$\ddot{f}_i + 2n_i\dot{f}_i + p_i^2 f_i = Q_i \sin p_i t,$$

where n_i = damping factor (depending on the number of resonant harmonic),

Q_i = reduced amplitude of driving force.

This equation is identical in meaning with Eq. (19.1). According to formula (19.4) the resonant amplitude is in this case

$$a^* = \frac{Q_i}{2_{np_i}}.$$

After calculating the resonant value a^* it is necessary to form the resonant values of the amplitudes of the angles of rotation; using formula (21.23) we obtain (leaving out in each line all terms except the ith term as small quantities of higher order)

$$\varphi_1 = a_{1i}a^*, \quad \varphi_2 = a_{2i}a^*, \quad \ldots, \quad \varphi_n = a_{ni}a^*.$$

Flexural Vibrations of Beams

The analysis of forced vibrations of multimass systems such as shown in Fig. 52*a* can be carried out by any one of the methods presented above (see p. 250). If the driving forces are assigned as one or two harmonics, the first method is preferred which can be applied without previously investigating free vibrations.

Suppose that the kth mass of a multimass system is subjected to the driving force

$$P(t) = P_k \sin \omega t, \tag{21.26}$$

where P_k = amplitude of force,
ω = its frequency.

Equations (7.28) will then involve additional terms corresponding to this disturbance

$$\left.\begin{aligned} y_1 &= -m_1\ddot{y}_1\delta_{11} - m_2\ddot{y}_2\delta_{12} - \ldots - m_n\ddot{y}_n\delta_{1n} + P_k\delta_{1k}\sin\omega t, \\ y_2 &= -m_1\ddot{y}_1\delta_{21} - m_2\ddot{y}_2\delta_{22} - \ldots - m_n\ddot{y}_n\delta_{2n} + P_k\delta_{2k}\sin\omega t, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= -m_1\ddot{y}_1\delta_{n1} - m_2\ddot{y}_2\delta_{n2} - \ldots - m_n\ddot{y}_n\delta_{nn} + P_k\delta_{nk}\sin\omega t. \end{aligned}\right\} \tag{21.27}$$

The steady-state part of the solution is

$$\left.\begin{aligned} y_1 &= a_1 \sin\omega t, \\ y_2 &= a_2 \sin\omega t, \\ &\cdots\cdots \\ y_n &= a_n \sin\omega t. \end{aligned}\right\} \tag{21.28}$$

The substitution of expressions (21.28) in Eqs. (21.27) leads to a system of equations which are used to determine the amplitudes of vibration $a_1, a_2, \ldots, a_n$

$$\left.\begin{aligned} a_1 &= m_1\omega^2\delta_{11}a_1 + m_2\omega^2\delta_{12}a_2 + \ldots + m_n\omega^2\delta_{1n}a_n + P_k\delta_{1k}, \\ a_2 &= m_1\omega^2\delta_{21}a_1 + m_2\omega^2\delta_{22}a_2 + \ldots + m_n\omega^2\delta_{2n}a_n + P_k\delta_{2k}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_n &= m_1\omega^2\delta_{n1}a_1 + m_2\omega^2\delta_{n2}a_2 + \ldots + m_n\omega^2\delta_{nn}a_n + P_k\delta_{nk}. \end{aligned}\right\} \tag{21.29}$$

All the coefficients (masses m_i, displacements δ_{ik}, frequency ω) are known in these equations and, to determine n amplitudes, it is necessary to solve the above system of n algebraic equations. This problem is simplified by using *Gauss' algorithm*. However, this algorithm is applicable only in cases where the coefficients of the unknowns possess the *reciprocity property*: the coefficient of the rth unknown in the sth line is equal to the coefficient of the sth unknown in the rth line. The system of equations (21.29) does not possess this property unless the masses m_i are equal; thus, the coefficient of a_2 in the first line ($m_2\delta_{12}\omega^2$) differs from the coefficient of a_1 in the second line ($m_1\delta_{21}\omega^2$). The reciprocity condition may be fulfilled by introducing new unknowns

$$A_1 = m_1a_1, \quad A_2 = m_2a_2, \quad \dots, \quad A_n = m_na_n.$$

The system of equations (21.29) then assumes the form

$$\begin{aligned}
&\left(\delta_{11}-\frac{1}{m_1\omega^2}\right)A_1+\delta_{12}A_2+\dots+\delta_{1n}A_n=-P_k\frac{\delta_{1k}}{\omega^2},\\
&\delta_{21}A_1-\left(\delta_{22}-\frac{1}{m_1\omega^2}\right)A_2+\dots+\delta_{2n}A_n=-P_k\frac{\delta_{2k}}{\omega^2},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\delta_{n1}A_1+\delta_{2n}A_2+\dots+\left(\delta_{nn}-\frac{1}{m_n\omega^2}\right)A_n=-P_k\frac{\delta_{nk}}{\omega_2}
\end{aligned}$$

satisfying the above condition since, according to Maxwell's theorem, $\delta_{ik} = \delta_{ki}$.

Example 25. The middle mass of a system (Fig. 135*a*) is subjected to a driving force $P \sin \omega t$. Determine the amplitudes of displacement of all masses

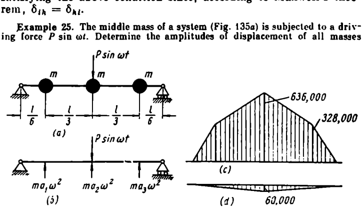


Fig. 135

and plot a dynamic bending moment diagram. The moment of inertia of the cross section of the beam is $I = 35{,}520\ \text{cm}^4$; the elastic modulus of the beam material

$E = 2.1 \times 10^6$ kgf/cm^2; the span of the beam $l = 400$ cm; the weight of the mass $Q = 4{,}000$ kgf; the mass $m = Q/g = 4.08$ kgf-sec^2/cm; the amplitude of the driving force $P = 600$ kgf; the frequency of the driving force $\omega = 100$ sec^{-1}.

To determine unit displacements we use the expressions given in Example 12 and calculate

$$k = \frac{l^3}{9 \times 1{,}296EI} = \frac{400^3}{9 \times 1{,}296 \times 2.1 \times 10^6 \times 35{,}520} = 7.38 \times 10^{-8}\ \text{cm/kgf},$$

$$\delta_{11} = \delta_{33} = 75k = 5.53 \times 10^{-6}\ \text{cm/kgf},$$

$$\delta_{22} = 243k = 17.92 \times 10^{-6}\ \text{cm/kgf},$$

$$\delta_{21} = \delta_{12} = \delta_{32} = \delta_{23} = 117k = 8.64 \times 10^{-6}\ \text{cm/kgf},$$

$$\delta_{13} = \delta_{31} = 51k = 3.76 \times 10^{-6}\ \text{cm/kgf}.$$

Equations (21.29) become

$$a_1 = 4.08 \times 5.53 \times 10^{-6} \times 10^4 a_1 + 4.08 \times 8.64 \times 10^{-6} \times 10^4 a_2 + \\ + 4.08 \times 3.76 \times 10^{-6} \times 10^4 a_3 + 600 \times 8.64 \times 10^{-6},$$

$$a_2 = 4.08 \times 8.64 \times 10^{-6} \times 10^4 a_1 + 4.08 \times 17.92 \times 10^{-6} \times 10^4 a_2 + \\ + 4.08 \times 8.64 \times 10^{-6} \times 10^4 a_3 + 600 \times 17.92 \times 10^{-6},$$

$$a_3 = 4.08 \times 3.76 \times 10^{-6} \times 10^4 a_1 + 4.08 \times 8.64 \times 10^{-6} \times 10^4 a_2 + \\ + 4.08 \times 5.53 \times 10^{-6} \times 10^4 a_3 + 600 \times 8.64 \times 10^{-6}.$$

Solving this system of equations, we obtain $a_1 = a_3 = -0.064$ cm and $a_2 = -0.128$ cm.

These amplitudes of vibration are more than 10 times larger than the corresponding static displacements

$$y_{1st} = y_{3st} = P_2\delta_{12} = 600 \times 8.64 \times 10^{-6} = 0.0052\ \text{cm},$$

$$y_{2st} = P_2\delta_{22} = 600 \times 17.92 \times 10^{-6} = 0.0107\ \text{cm},$$

this being due to the fact that the frequency of the disturbance is close to the natural frequency p_1; indeed, using the data obtained in Example 12, we find that the natural frequency differs only slightly from the frequency ω

$$p_1 = \frac{5.692}{l}\sqrt{\frac{EI}{ml}} = \frac{5.692}{400}\sqrt{\frac{2.1 \times 10^6 \times 35{,}520}{4.08 \times 400}} = 96.3\ \text{sec}^{-1}.$$

To plot a dynamic bending moment diagram it is necessary to consider the framework of the beam loaded by the amplitude values of the driving force and inertia forces $m_i a_i \omega^2$ developed by the concentrated masses (Fig. 135*b*). It should be noted that the inertia forces are opposite in phase to the given disturbance (since $\omega > p_1$).

Calculating gives

$$m_1 a_1 \omega^2 = -2{,}610\ \text{kg},$$

$$m_2 a_2 \omega^2 = -5{,}220\ \text{kg},$$

$$m_3 a_3 \omega^2 = -2{,}610\ \text{kg}.$$

The moment diagram corresponding to the above loading is shown in Fig. 135*c*. For comparison, Fig. 135*d* shows the bending moment diagram for the same beam loaded statically by a force of 600 kgf (both diagrams correspond to a loading additional to the purely static loading due to the weights *mg*).

Vibration Absorbers

Dynamic vibration absorbers. Revert to scheme of Fig. 133 and consider solution (21.5) in the case when the driving force acts only on the first mass, i.e., $P_1 \neq 0$, $P_2 = 0$

$$a_1 = \frac{P_1(k_2 - m_2\omega^2)}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2},$$

$$a_2 = \frac{P_1 k_2}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}.$$

From these formulas it is seen that in the particular case when

$$k_2 - m_2\omega^2 = 0 \tag{21.30}$$

the amplitudes a_1 and a_2 take the following values

$$a_1 = 0, \quad a_2 = -\frac{P_1}{k_2}, \tag{21.31}$$

i.e., the first mass remains *stationary* even though it is the one to which the driving force is applied (antiresonance of the first mass).

This curious result is used as the basis in the design of a *dynamic vibration absorber*. Let, for instance, a system (Fig. 136*a*) be subjected to a driving force $P \sin \omega t$. In order to "damp" the vibrations of this system, it is sufficient to attach to it an additional elastically mounted mass (Fig. 136*b*) imposing condition (21.30) on the parameters of the added system. Then the vibrations of the main mass will vanish and the amplitude of vibration of the additional mass will be determined by the second formula of (21.31). The attachment of an additional mass to a main structure changes the number of degrees of freedom of the system and accordingly the number of natural frequencies. If a system with no dynamic absorber had one degree of freedom (Fig. 136*a*) and one natural frequency

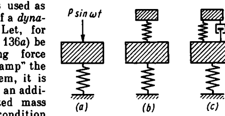


Fig. 136

$$p_0 = \sqrt{\frac{k_1}{m_1}},$$

then, on attaching the absorber, the number of degrees of freedom becomes two, and the frequencies are determined by Eq. (7.11*b*). Denote the frequency of the absorber with the main mass stationary by

$$\bar{p}_0 = \sqrt{\frac{k_2}{m_2}}.$$

Equations (7.11*b*) can then be rewritten as

$$\frac{p^4}{\bar{p}_0^4} - \frac{p^2}{\bar{p}_0^2}\left(\frac{p_0^2}{\bar{p}_0^2} + 1 + \mu\right) + \frac{p_0^2}{\bar{p}_0^2} = 0, \tag{21.32}$$

where $\mu = m_2/m_1$.

Consider *the case of resonance* when the frequency p_0 of the system with no absorber equals the disturbing frequency ω. Since, according to formula (21.30), a properly designed absorber must satisfy the condition

$$\bar{p}_0^2 = \frac{k_2}{m_2} = \omega^2,$$

the resonant vibrations of the main mass can be completely damped out if

$$\bar{p}_0^2 = p_0^2.$$

In this case Eq. (21.32) takes the form

$$\frac{p^4}{p_0^4} - \frac{p^2}{p_0^2}(2+\mu) + 1 = 0$$

and has the following solution

$$p_{1,2}^2 = p_0^2\left(1 + \frac{\mu}{2} \pm \sqrt{\mu + \frac{\mu^2}{4}}\right). \tag{21.33}$$

Thus, inclusion of the absorber splits the natural frequency in two; if, for instance, $\mu = 0.2$ (the additional mass is one-fifth of the main mass), we find from expression (21.33): $p_1 = 0.8p_0$, $p_2 = 1.25p_0$.

Under operating conditions, the disturbing frequency may sometimes vary over a more or less wide range. If, in the above example, the frequency ω is reduced by 20 per cent ($\omega = 0.8p_0$ instead of $\omega = p_0$), resonance occurs with the first frequency p_1. Such an absorber may do only harm.

Therefore, installation of a dynamic vibration absorber may be recommended only in cases where the driving force has a stable frequency. The performance of an absorber may be greatly improved by introducing inelastic resistances (Fig. 136*c*), but in this case, too, installation of a dynamic absorber is not advisable if the frequency of disturbance varies over a wide range (foundations for piston compressors, internal-combustion engines, etc.).

The foregoing principle may also be applied for damping torsional vibrations. A small additional disk, if properly tuned, may serve as a dynamic damper of torsional vibrations of a two-mass system (Fig. 137).

If it is necessary to eliminate the vibrations of the main system subjected to a disturbance of frequency ω, then, as follows from Eq. (21.30), the natural frequency of the absorber $\bar{p}_0$ when attached to a fixed point must be equal to the frequency ω, i.e.,

$$\sqrt{\frac{k}{J_p}}=\omega,$$

where k = torsional stiffness of absorber shaft,

J_p = mass moment of inertia of absorber disk.

It is clear that an additional elastically attached disk tuned to one particular frequency will damp the vibrations of this frequency only and may be inefficient or even cause resonance at other frequencies of disturbance. This is particularly important for shafts of internal-combustion engines since the frequency of disturbance varies directly as the number of revolutions.

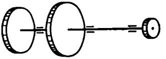


Fig. 137

In such cases it is therefore desirable to provide an absorber with *follow-up tuning* so that as the frequency of disturbance varies the natural frequency of the absorber will vary accordingly. Since the elastic attachment of an additional disk cannot ensure follow-up tuning, *pendulum absorbers* are used to damp vibrations of rotating shafts.

As shown above (see p. 36), a pendulum suspended from a rotating disk has a natural frequency proportional to the angular velocity

$$\bar{p}_0=\omega\sqrt{\frac{R}{l}}, \qquad (21.34)$$

where R = distance from disk centre to point of suspension of pendulum,

l = length of pendulum.

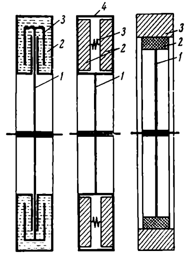
On the other hand, in internal-combustion engines the frequency of any harmonic of the disturbance is also proportional to the speed of rotation ω. Therefore, such a pendulum can serve as a dynamic absorber of vibrations produced by *one* particular harmonic at any speed of rotation.

The most dangerous harmonic must be eliminated in the first place. Let it be decided, for instance, to eliminate the vibrations produced by a harmonic of the disturbance having a frequency 3ω. Then, from formula (21.34) it follows that $\omega\sqrt{R/l}=3\omega$, i.e.,

$R/l = 9$. To damp the vibrations produced by any other harmonic, a different ratio R/l would be required.

Since the dimension R is limited, the length l is very small; this makes it difficult to design a pendulum such as that of Fig. 20.

This difficulty can be overcome, in particular, by using a bifilar-type pendulum (see Fig. 21). In this arrangement, according to formula (4.17), the design length l may be made as small as desired by properly reducing the difference between the diameters of the rollers and holes.



(a) (b) (c)

Fig. 138

Although the foregoing theory presupposes a linear type of vibration, the linearity of a system should not be regarded as the condition for effective damping action; on the contrary, the nonlinearity inherent in large oscillations of a pendulum aids in damping oscillations.

Vibration dampers. An efficient means of reducing objectionable vibrations is offered by devices providing intensive energy dissipation. Thus, there exist various schemes of torsional vibration dampers: fluid friction dampers (Fig. 138*a*), dry friction dampers (Fig. 138*b*) and hysteresis dampers (Fig. 138*c*). In all the three diagrams, the disks whose torsional vibrations are to be damped are denoted by the figure *1*.

In the scheme of Fig. 138*a*, there is an annular cavity *2* filled with fluid and the damping action is produced by elements *3* attached to the disk and moving with respect to the fluid filling cavity *2*.

In the dry friction damper (Fig. 138*b*), a system of springs *3* is used to press disks *2* against the annular cavity walls *4*. If the pressure is too great for slipping, the disks will move with disk *1* and no energy will be dissipated; if the pressure is very slight, the dissipation will be insignificant because the friction force between the disks and the walls is small. Hence, the design of the damper requires a special choice of the optimum pressure on the disks.

In the scheme of Fig. 138*c*, the disk carries a damping ring *2* with a press-fitted band *3*; when the disk is vibrating, intensive energy dissipation occurs in the ring gasket *2*, which is made of a material with high internal friction.

In some cases detrimental vibrations can be reduced or completely eliminated by introducing *shock dampers*; in these dampers, energy is dissipated due to collisions of weights, which have some freedom of movement within the housing, with special stops.

22. Linear Systems with Distributed Parameters

The three methods presented above (see p. 250) for solving forced vibration problems for systems of several degrees of freedom are also suitable for the analysis of vibrations of systems with distributed mass. The choice of method is governed by the particular driving forces; for a harmonic disturbance the first method is more convenient and for an arbitrarily given disturbance, the third one.

Longitudinal Vibrations of Bars

Harmonic disturbance; direct solution. Consider the case when a bar is subjected to a single concentrated longitudinal force varying according to a harmonic law

$$P = P_0 \sin \omega t. \tag{22.1}$$

The steady-state forced vibrations occur with the impressed frequency and hence follow the law

$$u(x,\ t) = U(x) \sin \omega t, \tag{22.2}$$

where $U(x)$ is a function of the abscissa which is to be determined (the *mode of forced vibration*). For an element of the bar (see Fig. 61) we obtain Eq. (8.3); substituting expression (22.2), we arrive at an ordinary differential equation for the function $U(x)$

$$U'' + \frac{\omega^2}{k^2} U = 0, \tag{22.3}$$

where k^2 is defined by formula (8.4).

Equation (22.3) differs from Eq. (8.6) for the mode of free vibration in that the frequency is known in advance. Similarly to expression (8.8), we write the solution of Eq. (22.3) as

$$U = C \sin \frac{\omega}{k} x + D \cos \frac{\omega}{k} x. \tag{22.4}$$

The constants C and D must be determined from the boundary conditions which are formulated as follows.

Fixed end of a bar. In this case $u = 0$ at any time t; this requires that at this section $U = 0$.

Driving force (22.1) *at the end of a bar.* The applied force must be equal to the longitudinal force at the end section. According to formula (8.1) we have

$$N = EA\frac{\partial u}{\partial x} = EAU' \sin \omega t. \tag{22.5}$$

Equating expressions (22.1) and (22.5), we obtain the boundary condition

$$U' = \frac{P_0}{EA}. \tag{22.6}$$

Free end of a bar. According to formula (22.6) $U' = 0$.

Concentrated mass m_0 at the end of a bar. The inertia force developed by the mass $-m_0\ddot{u}_0 = m_0\omega^2 U_0 \sin \omega t$ (here u_0 and U_0 pertain to the point of attachment of the mass m_0) must be equal to the longitudinal force $N_0 = EA\frac{\partial u}{\partial x} = EAU_0' \sin \omega t$. Consequently,

$$\frac{m_0\omega^2}{EA}U_0 = U_0'.$$

Example 26. Determine the amplitude of vibration of the end of a bar to which a driving force $P = P_0 \sin \omega t$ is applied. The other end of the bar is fixed.

Placing the origin of co-ordinates at the fixed end, we have the boundary conditions

$$U = 0 \quad \text{at} \quad x = 0,$$

$$U' = \frac{P_0}{EA} \quad \text{at} \quad x = l.$$

Substituting in solution (22.4), we obtain

$$C = \frac{P_0}{EA\frac{\omega}{k}\cos\frac{\omega}{k}l}, \quad D = 0.$$

Consequently, the amplitude of vibration of the end of the bar is

$$U(l) = \frac{P_0 k}{EA\omega}\tan\frac{\omega l}{k}. \tag{22.7}$$

At a very low frequency (i.e., when the force P varies slowly) $\tan(\omega l/k) \approx \omega l/k$ and formula (22.7) reduces to the expression for the static displacement

$$U(l) = \frac{P_0 l}{EA}.$$

When $\omega l/k = \pi/2$, i.e., when $\omega = \pi k/2l$, the amplitude $U(l)$ becomes infinite which corresponds to resonance, the frequency ω being equal to the lowest natural frequency. Resonance with higher frequencies corresponds to the disturbing frequencies

$$\omega = (2n + 1)\frac{\pi k}{2l} \qquad (n = 2, 3, \ldots).$$

When $\omega l/k = n\pi$ $(n = 1, 2, \ldots)$ the end section is stationary (antiresonance).

Example 27. Solve the same problem on the assumption that the end of the bar carries a concentrated mass m_0.

Using the previous co-ordinate system, we have the boundary conditions

$$U=0 \quad \text{at} \quad x=0,$$

$$U'=\frac{P_0}{EA}+\frac{m_0\omega^2}{EA}U \quad \text{at} \quad x=l.$$

The second boundary condition expresses that the longitudinal force at the right end of the bar is equal to the sum of the driving force and the inertia force of the end mass. Substituting the first of the conditions in expression (22.4), we find $D=0$, from the second condition we obtain

$$C=\frac{P_0}{EA\frac{\omega}{k}\cot\frac{\omega l}{k}-m_0\omega^2}.$$

The amplitude of vibration of the end of the bar is

$$U(l)=\frac{P_0\sin\frac{\omega l}{k}}{EA\frac{\omega}{k}\cot\frac{\omega l}{k}-m_0\omega^2}.$$

If the bar has a section varying *according to a step law*, solution (22.4) must be written for each of the portions separately

$$U_1=C_1\sin\frac{\omega}{k}x_1+D_1\cos\frac{\omega}{k}x_1,$$

$$U_2=C_2\sin\frac{\omega}{k}x_2+D_2\cos\frac{\omega}{k}x_2,$$

$$\dots\dots\dots\dots\dots\dots$$

$$U_n=C_n\sin\frac{\omega}{k}x_n+D_n\cos\frac{\omega}{k}x_n,$$

where 1, 2, . . . are the numbers of the corresponding portions. The constants C_i and D_i are determined from two conditions at the ends of the bar and $2(n-1)$ *matching conditions*. The latter express the equality of displacements and longitudinal forces at the boundary of two portions

$$U_{i-1}(l_{i-1})=U_i(0),$$

$$(EAU')_{i-1}(l_{i-1})=(EAU')_i(0).$$

The same procedure should be followed in cases where a driving force is applied at several intermediate sections.

Consider the case when the external load is continuously distributed along the length of the bar

$$q(x,t)=q_0(x)\sin\omega t. \tag{22.8}$$

Referring to Fig. 61, we write the equation of motion taking into account an elementary external load $q_0(x)\,dx\sin\omega t$; using relation (22.2) and assuming $EA = \text{constant}$, we obtain

$$U'' + \frac{\omega^2}{k^2} U = \frac{q_0}{FA}.$$

The general solution of this differential equation depends on the particular right-hand side. Thus, if $q_0 = \text{constant}$,

$$U = C \sin \frac{\omega}{k} x + D \cos \frac{\omega}{k} x + \frac{q_0}{\rho A \omega^2}.$$

In the general case

$$U = C \sin \frac{\omega}{k} x + D \cos \frac{\omega}{k} x + \frac{k}{EA\omega} \int_0^x q_0(\xi) \sin \frac{\omega}{k} (x - \xi)\, d\xi. \quad (22.9)$$

If the section of the bar *varies continuously*, the basic differential equation is written as

$$k^2 \frac{\partial}{\partial x} \left(A \frac{\partial u}{\partial x} \right) = A \frac{\partial^2 u}{\partial t^2}$$

(no distributed external load). After substituting expression (22.2), the equation becomes

$$(AU')' + \frac{\omega^2}{k^2} AU = 0; \quad (22.10)$$

it has variable coefficients and cannot be solved in closed form in the general case. Therefore, approximate methods should be used, such as the Bubnov-Galerkin method. To a first approximation, assuming a suitable function $U_1(x)$ for $U(x)$, we obtain an equation of the form

$$\int_0^l \left[(AU_1')' + \frac{\omega^2}{k^2} AU_1 \right] U_1\, dx = 0. \quad (22.11)$$

The function U_1 must satisfy the boundary conditions; at least one of these conditions depends on the amplitude of the driving force [for instance, condition (22.6)]. Therefore, the function U_1 must be assigned so as to satisfy both boundary conditions of the problem and, in addition, to have one undetermined parameter which will be found from Eq. (22.11).

Solution in terms of eigenfunctions. In the general case, the external load is arbitrarily distributed along the length and is an arbitrary function of time

$$q = q(x, t).$$

In particular, the load may vary in time according to a law common to all points

$$q = q_0(x) H(t). \quad (22.12)$$

Taking into account an elementary external load $q\,dx$ and assuming $EA=\text{constant}$, we obtain from Fig. 61

$$EA\frac{\partial^2 u}{\partial t^2}-\rho A\frac{\partial^2 u}{\partial x^2}=q(x,\,t). \tag{22.13}$$

Further the load $q(x,\,t)$ and the displacement $u(x,\,t)$ are represented in the form of series in eigenfunctions of the corresponding free-vibration problem [see formula (8.8)]

$$\left.\begin{aligned} q&=X_1(x)\,S_1(t)+X_2(x)\,S_2(t)+\ldots,\\ u&=X_1(x)\,T_1(t)+X_2(x)\,T_2(t)+\ldots. \end{aligned}\right\} \tag{22.14}$$

To determine the functions of time $S_i(t)$, we multiply both sides of Eq. (22.13) by $X_i(x)$, and integrate the result over the total length of the bar. Upon integration all the terms on the right-hand side vanish except the ith term (due to the orthogonality property of the eigenfunctions) and the following formula for $S_i(t)$ is obtained

$$S_i(t)=\frac{\int\limits_0^l q(x,\,t)\,X_i(x)\,dx}{\int\limits_0^l X_i^2(x)\,dx}. \tag{22.15}$$

If the load varies according to law (22.12), then

$$S_i(t)=H(t)\,\frac{\int\limits_0^l q_0(x)\,X_i(x)\,dx}{\int\limits_0^l X_i^2(x)\,dx}, \tag{22.15a}$$

i.e., the functions $S_i(t)$ for all numbers i differ only in scale.

If the bar is loaded by concentrated forces $P_a(t)$, $P_b(t)$. . . . at sections with abscissas a, b, . . . formula (22.15) becomes

$$S_i(t)=\frac{P_a(t)\,X_i(a)+P_b(t)\,X_i(b)+\ldots}{\int\limits_0^l X_i^2(x)\,dx}. \tag{22.16}$$

The determination of the functions $T_i(t)$ is based on the fact that each term on the right-hand side of the upper series of (22.14) produces the motion defined by the corresponding term of the lower series of (22.14). Therefore, we can substitute in Eq. (22.13)

$$q=X_i(x)\,S_i(t),\quad u=X_i(x)\,T_i(t). \tag{22.17}$$

We then obtain the equation

$$EAX_i''T_i-\rho AX_i\ddot{T}_i=X_iS_i$$

or after dividing by $\rho A X_i T_i$

$$k^2\frac{X_i''}{X_i}=\frac{\ddot{T}_i}{T_i}+\frac{S_i}{\rho A T_i}.$$

According to the first formula of (8.5*a*), the left-hand side of the above equality is constant and equal to $-p_i^2$. Consequently, the right-hand side equals the same quantity; hence

$$\ddot{T}_i+p_i^2T=\frac{S_i}{\rho A}.$$

As has been shown previously (see p. 197), the solution of this equation is of the form

$$T_i=\frac{1}{\rho A p_i}\int_0^t S_i(\tau)\sin p_i(t-\tau)\,d\tau.$$

This formula solves the problem as it provides a means of forming the second of the sums (22.14).

If the load follows law (22.12), we can write, taking into account expression (22.15*a*),

$$T_i=\frac{\int_0^l q_0(x)\,X_i(x)\,dx}{\rho A P_i\int_0^l X_i^2(x)\,dx}\int_0^t H(\tau)\sin p_i(t-\tau)\,d\tau.$$

In the case of an arbitrary periodic function $H(\tau)$ use should be made of the methods presented above (see pp. 221 to 226). Of course, in the simplest case (22.8) it is easier to seek the solution in the form of (22.2) determining $U(x)$ by means of expression (22.9).

Flexural Vibrations of Beams

Harmonic disturbance; direct solution. Consider the case when the impressed load is assigned as a concentrated force

$$P=P_0\sin\omega t \tag{22.18}$$

or a combination of several loads of this type with the same frequency. The solution for deflections will be sought in the form

$$y(x,\ t)=Y(x)\sin\omega t \tag{22.19}$$

reducing the problem to the determination of the mode of vibration (amplitudes curve) $Y(x)$

In the case of m = constant, substituting expression (22.19) in Eq. (8.19), we obtain, similarly to Eq. (8.22),

$$Y^{\mathrm{IV}} - \frac{m\omega^2}{EI} Y = 0. \tag{22.20}$$

The solution of the differential equation (22.20) is represented by an expression similar to (8.24)

$$Y = C_1 S + C_2 T + C_3 U + C_4 V, \tag{22.21}$$

where S, T, U and V are Krylov's functions (8.25) in which expression (8.23) should be replaced by

$$k = \sqrt[4]{\frac{m\omega^2}{EI}}\,.$$

To determine the constants C_1, C_2, C_3 and C_4 appearing in the general solution (22.21) we have to use the boundary conditions. In addition to the foregoing [see formulas (8.28*a*) to (8.31)] it is necessary to consider at least two more cases here.

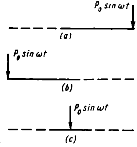


Fig. 139

The driving force $P_0 \sin \omega t$ is applied *at the end of the beam* (Fig. 139*a* and *b*). Then the shearing force must be equal to this force

$$Q = EIy''' = EIY''' \sin \omega t$$

and the boundary condition becomes

$$Y''' = \pm \frac{P_0}{EI}$$

(the upper sign corresponds to a force applied at the right end and the lower sign to a force applied at the left end). Besides, there must be $Y'' = 0$.

The driving force $P_0 \sin \omega t$ is applied *at an intermediate section of the beam* (Fig. 139*c*). The following four matching conditions must be fulfilled at this section

$$Y(a_-) = Y(a_+), \qquad Y'(a_-) = Y'(a_+),$$

$$Y''(a_-) = Y(a_+), \qquad Y'''(a_-) - Y'''(a_+) = \frac{P_0}{EI},$$

where a is the abscissa of the section at which the driving force is applied. The subscripts — and + correspond to sections just to the left or to the right of the section a.

The first three conditions express the continuity of the deflection, the angle of rotation of the section and the bending moment at the point of application of the driving force; the fourth condition expresses the discontinuity in the shearing force at the section in question by the amount P_0.

For a larger number of driving forces, the method of initial parameters may be used to advantage.

Example 28. Determine the amplitudes curve for a cantilever beam of length l if the driving force (22.18) is applied to its free right end.

The boundary conditions are

$$Y(0)=0, \qquad Y'(0)=0,$$

$$Y''(l)=0, \qquad Y'''(l)=\frac{P_0}{EI}.$$

The first two conditions give $C_1=C_2=0$, the third and fourth conditions lead to the equations

$$k^2(C_3S_l+C_4T_l)=0,$$

$$k^3(D_3V_l+C_4S_l)=\frac{P_0}{EI}.$$

Hence

$$C_3=-\frac{T_l}{k^3EI\,(S_l^2-T_lV_l)}P_0.$$

$$C_4=\frac{S_l}{k^3EI\,(S_l^2-T_lV_l)}P_0.$$

Substituting in expression (22.21), we find

$$Y=\frac{P_0(S_lV_x-T_lU_x)}{k^3EI\,(S_l^2-T_lV_l)}. \qquad (22.22)$$

The amplitude of the deflection of the end of the cantilever is

$$Y(l)=\frac{P_0(S_lV_l-T_lU_l)}{k^3EI\,(S_l^2-T_lV_l)}.$$

Substituting Krylov's functions, we obtain

$$Y(l)=\frac{P_0(\sinh kl\cos kl-\cosh kl\sin kl)}{k^2EI\,(\cosh kl\cos kl+1)}.$$

It is seen that when

$$\cosh kl\cos kl+1=0$$

resonance occurs; this case corresponds to the coincidence of the frequency of the driving force with one of the natural frequencies. If, however, the following equality is fulfilled

$$\tan kl=\tanh kl,$$

the end of the cantilever will be stationary (antiresonance).

Solution in terms of eigenfunctions. In the general case when the impressed transverse load is given by an arbitrary law

$$q=q(x,\ t)$$

the differential equation assumes the form

$$\frac{\partial^2 y}{\partial t^2}+\frac{EI}{m}\,\frac{\partial^4 y}{\partial x^4}=\frac{q}{m} \tag{22.23}$$

and differs from Eq. (8.19) by the presence of the right-hand side.

As before, we represent $q(x, t)$ in the form of a series

$$q(x, t)=X_1(x)\,S_1(t)+X_2(x)\,S_2(t)+\ldots \tag{22.24}$$

and seek the solution for the deflection also in the form of a series

$$y(x, t)=X_1(x)\,T_1(t)+X_2(x)\,T_2(t)+\ldots\ . \tag{22.25}$$

To determine the functions of time $S_i(t)$, we multiply both sides of equality (22.24) by $X_i(x)$ and integrate the result over the total length of the beam; due to the orthogonality of the eigenfunctions there remains only one term on the right-hand side, which corresponds to the number i, so that

$$S_i(t)=\frac{\int\limits_0^l q(x, t)\,X_i(x)\,dx}{\int\limits_0^l X_i^2(x)\,dx}\,. \tag{22.26}$$

This formula is identical in form with formula (22.15) derived above for the case of longitudinal vibrations, but in expression (22.26) $X_i(x)$ represent the natural modes in the problem of free vibration of the beam ("*beam functions*"). Therefore, formula (22.16) pertaining to the case of concentrated driving forces is also valid here.

Taking into account that each term of series (22.24) produces the motion defined by the corresponding term of series (22.25), we can write Eq. (22.23) as

$$X_i\ddot{T}_i+\frac{EI}{m}\,X_i^{\mathrm{IV}}T_i=\frac{X_iS_i}{m}\,.$$

Dividing both sides by X_iT_i, we obtain

$$-\frac{EI}{m}\,\frac{X_i^{\mathrm{IV}}}{X_i}=\frac{\ddot{T}_i}{T_i}-\frac{S_i}{mT_i}\,.$$

The left-hand side of this equality is equal to $-p_i^2$; therefore,

$$\frac{\ddot{T}_i}{T_i}-\frac{S_i}{mT_i}=-p_i^2.$$

Hence we obtain the differential equation for T_i

$$\ddot{T}_i+p_i^2T_i=\frac{S_i}{m}\,.$$

The general solution of this equation is

$$T_i(t) = \frac{1}{mp_i} \int_0^t S_i(\tau) \sin p_i (t-\tau)\, d\tau. \tag{22.27}$$

Example 29. Construct the solution for the case when the end of a cantilever beam is subjected to an arbitrarily given concentrated force $P_i(t)$.

In this case, according to expression (22.26)

$$S_i(t) = \frac{X_i(l)}{\int_0^l X_i^2(x)\, dx} P_i(t)$$

and formula (22.27) becomes

$$T_i(t) = \frac{X_i(l)}{mp_i \int_0^l X_i^2(x)\, dx} \int_0^t P_i(\tau) \sin p_i (t-\tau)\, d\tau.$$

Using the last expression we can obtain the solution for any law $P_i(t)$; in the case of a periodic action (particularly, periodic shocks) use should be made of the method presented on pp. 223-224.

The foregoing method provides a means for obtaining the solution in the case of a variable section if the natural modes X_i and the natural frequencies p_i are found in advance.

CHAPTER V

PARAMETRIC VIBRATIONS

23. Basic Equation

In some cases the parameters of a mechanical system (i.e., its stiffness or mass) do not remain unchanged but are some given functions of time (usually periodic functions).

If the equilibrium state of such a system is disturbed, peculiar vibrations will result: on the one hand, they cannot be called free vibrations since the system is subjected to an external action in the form of the variation of its stiffness and, on the other hand, they are not forced vibrations as the external action is not produced by a given force. These vibrations are referred to as *parametric* and, depending on the properties of the system and the type of variation of its parameters, they may have bounded or increasing amplitudes; the latter case, which is obviously a dangerous one, is called *parametric resonance*.

The Simplest System

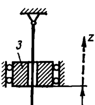


Fig. 140

Consider the system shown in Fig. 140. As is seen, the concentrated mass 1 is attached to the end of a weightless rod 2, the freedom of displacement of the rod being additionally restrained by a collar 3 removed a distance l from the lower end of the rod. Write the equation of free vibration of the mass assuming the vibrations to occur in the plane of the drawing. If at the current instant t the displacement of the mass is x, the elastic restoring force of the rod is $-kx$ and the equation of motion of the mass is of the form

$$-kx = m\ddot{x}, \tag{23.1}$$

where k is the spring constant of the system.

Collar *3*, if sufficiently long, provides practically complete restraint of the lower portion of the rod and the constant k can be determined from the known formula $k = 3EI/l^3$. It is assumed here that the rod has a constant cross section of moment of inertia I; E is the elastic modulus of the rod material. Thus, the differential equation (23.1) takes the form

$$\ddot{x} + \frac{3EI}{ml^3} x = 0. \tag{23.2}$$

If the distance l is constant, Eq. (23.2) describes free vibrations of the mass about its central position, the quotient $3EI/ml^3$ representing the square of the natural frequency of vibration.

Suppose now that collar *3* slides along rod *2* according to a given law

$$z = A \cos \omega t,$$

i.e., it executes harmonic vibrations with amplitude A and circular frequency ω about its central position l. In this case the spring constant is a *function of time*

$$k = \frac{3EI}{(l+z)^3} = \frac{3EI}{(l + A \cos \omega t)^3} \tag{23.3}$$

and the differential equation (23.2) becomes an *equation with variable coefficients*

$$\ddot{x} + \frac{3EI}{m(l + A \cos \omega t)^3} x = 0, \tag{23.4}$$

this being typical of systems with parametric excitation of vibrations.

Mathieu's Equation

There exist many other mechanical systems subjected to parametric excitation. Several examples given below demonstrate that in most practically important cases the differential equation governing parametric vibrations can be reduced to the *standard form*

$$\frac{d^2x}{d\tau^2} + (a - 2q \cos 2\tau) x = 0, \tag{23.5}$$

where a and q are constants.

By way of illustration we revert to the mechanical system shown in Fig. 140 and assume that the amplitude A of vibration of the collar is very small in comparison with the length l; then, instead of expression (23.3) we obtain an approximation

$$k = \frac{3EI}{(l + A \cos \omega t)^3} \cong \frac{3EI}{l^3 + 3Al^2 \cos \omega t} \cong \frac{3EI}{l^3}\left(1 - \frac{3A}{l} \cos \omega t\right) \tag{23.6}$$

and the differential equation (23.4) becomes

$$\ddot{x}+\frac{3EI}{ml^3}\left(1-\frac{3A}{l}\cos\omega t\right)x=0. \tag{23.7}$$

Passing now to the "dimensionless" time τ

$$2\tau=\omega t,$$

we have

$$\frac{d^2x}{dt^2}=\frac{\omega^2}{4}\frac{d^2x}{d\tau^2}$$

and the differential equation (23.7) assumes the standard form (23.5) with

$$\left.\begin{aligned} a&=\frac{12EI}{m\omega^2l^3},\\ q&=\frac{18AEI}{m\omega^2l^4}.\end{aligned}\right\} \tag{23.8}$$

The transformations of this kind are typical for cases of small depth of pulsation of the variable parameter of a system.

Revert now to the basic equation of the problems here considered (23.5) which is called *Mathieu's equation*. The solutions of Mathieu's equation are vibratory ones and depend entirely on the particular values of the parameters a and q. In some cases, to a given combination of a and q correspond vibrations bounded in amplitude and in others, vibrations with increasing amplitudes. As a matter of fact, further details of the vibration are of little significance since it is the tendency of the vibratory process that is of prime importance: if the amplitudes remain bounded, the system is *stable*; otherwise, parametric resonance occurs and the system is *unstable*.

The results of solving Mathieu's equation for two different combinations of a and q are shown in Fig. 141 (these solutions have been obtained with the use of an electronic analogue device). Although the parameter q of the system is the same ($q = 0.1$) in both cases, the vibrations differ greatly because of the difference between the values of the parameter a ($a = 1$, $a = 1.2$); in the first case they increase, i.e., the system is unstable, and in the second case they remain bounded, i.e., the system is stable.

For practical purposes the boundaries between the regions of stable and unstable solutions are of the greatest importance. This question is well studied, the final results being represented as a diagram plotted in the plane of parameters a and q. It is called the *Ince-Strutt diagram*; Figure 142 shows a part of the Ince-Strutt diagram pertaining to small values of the parameter q. To each particular system characterized by parameters a and q corresponds a point of co-ordinates a, q on the Ince-Strutt diagram (*generic point*). If the generic point is within the shaded areas of the diagram,

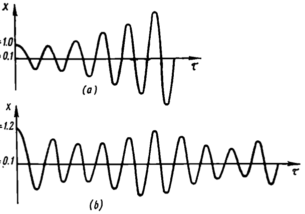


Fig. 141

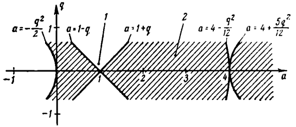


Fig. 142

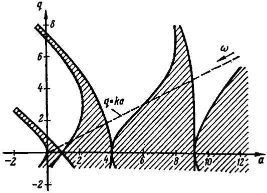


Fig. 143

the system is stable; to unstable systems correspond generic points in blank areas.

By way of example, in the diagram are indicated points *1* and *2* which correspond to the parameters $a_1 = 1$, $q_1 = 0.1$; $a_2 = 1.2$, $q_2 = 0.1$ (the solutions of Mathieu's equation for these cases are given in Fig. 141). Point *1* is in the blank zone (instability) and the vibrations occur with increasing amplitudes (Fig. 141*a*). Point *2* is within the shaded zone; it corresponds to the motion with a bounded amplitude (Fig. 141*b*).

The complete Ince-Strutt diagram is represented in Fig. 143. As is seen, stability regions alternate with instability regions in the plane of parameters a, q, the widest and therefore the most important instability region containing the point $a = 1$, $q = 0$. The Ince-Strutt diagram eliminates the necessity of performing any of the operations involved in solving Mathieu's equation. It is sufficient to set up this equation, i.e., to find the values of the parameters a and q of the system; after that the diagram gives at once the answer to the question as to whether the system is stable or unstable.

Let us see how the properties of parametric vibrations vary when the exciting frequency is gradually changed. Reverting, by way of example, to expressions (23.8), we note that, as the frequency increases, both parameters, a and q, decrease in proportion. Since the ratio between the two parameters remains constant, the successive states of the system are determined by generic points on a dashed ray $q = ka$ passing through the origin. Figure 143 shows clearly the alternation of stable and unstable states for increasing values of the exciting frequency.

24. Cases of Periodic Variation of Stiffness

These cases are common in all kinds of systems. Consider several examples.

Figure 144*a* shows a system the elastic part of which is a toothed (splined) shaft *1*. The lower end of the shaft carries disk *2*. The shaft is connected to a toothed (splined) heavy collar *3* which can slide along the axis of the shaft and execute harmonic vibrations in the vertical direction. In this system it is possible to excite not only flexural but also torsional vibrations. Let the free length of the shaft at the current instant t be

$$l = l_0 + A\cos\omega t.$$

The torsional stiffness of the shaft is then

$$k = \frac{GI_p}{l} = \frac{GI_p}{l_0 + A\cos\omega t}. \qquad (24.1)$$

If the amplitude of vibration A is considerably smaller than the average value of the length l_0, expression (24.1) can be represented as

$$k=\frac{GI_p}{l_0}\left(1-\frac{A}{l_0}\cos\omega t\right),$$

which is identical in form with expression (23.6). Therefore, the torsional vibrations of the system under consideration are also described by Mathieu's equation (23.5) with

$$a=\frac{4GI_p}{\omega^2 l_0},$$

$$q=\frac{2AGI_p}{\omega^2 l_0^2}.$$

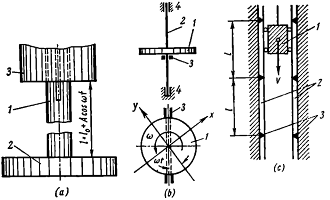


Fig. 144

Consequently, this system may be in a state of parametric resonance under certain conditions defined by the Ince-Strutt diagram. Of course, this is also true in cases where the axis of the shaft does not coincide with the vertical.

Another example of a system with periodic variation of stiffness is presented in Fig. 144*b*. The system contains a disk fixed at the middle of a vertical shaft *2*. Over part of its length the shaft has a cross section with different principal moments of inertia; for this reason the rigidity of the shaft is not the same in the two principal directions x and y. Guides *3* fix a plane in which the shaft can deflect. Therefore, as the shaft rotates in bearings *4*, its flexural rigidity in this plane varies periodically and parametric excitation of vibrations is possible.

Note that if there are no guides, then, as pointed out above (see p. 173), the shaft is unstable over the entire range of speeds

$$\sqrt{\frac{k_1}{m}} < \omega < \sqrt{\frac{k_2}{m}} .$$

An instructive example of parametric excitation of vibration is provided by the system shown in Fig. 144*c*. A mine cage *1* moves with a constant velocity along vertical guides *2* which are fixed on sleepers *3*. In this system the transverse stiffness, which defines the elastic restoring force during transverse vibrations of the cage, is variable: if the cage is level with any one of the pairs of sleepers, this stiffness reaches a maximum; if, however, the cage is located opposite the middle of the free span of the guides, the stiffness is minimum. The frequency of the stiffness variation depends on the distance between the sleepers and the velocity of the cage

$$\omega = \frac{2\pi v}{l} .$$

It is clear that there exist a number of "forbidden" ranges of velocity v corresponding to the conditions of parametric resonance. These considerations are fully supported by experimental evidence.

26. Cases of Periodic Variation of Parametric Loads

A simple example of a system of the type here considered is presented in Fig. 145*a*. Mass *1* is attached to the upper end of a vertical perfectly rigid rod *2*; in its lower portion the rod has a support which provides elastic resistance to rotation ("elastic hinge"). The upper end of the rod is acted on by a vertical force P. This system results from a simplified schematization of a real rod having a distributed mass and elasticity.

The force P is a *parametric load* and, if it is constant in time, its critical value can be found by Euler's method. Let φ be the angular displacement of the rod from the vertical and k the spring constant of the elastic hinge. The restoring moment (the moment of the elastic hinge) is then $-k\varphi$ and the equation of equilibrium of the rod in the displaced position is

$$Pl\varphi - k\varphi = 0. \tag{25.1}$$

From the condition $\varphi \neq 0$ we find that the displaced position of equilibrium is possible if the force P is equal to

$$P_{cr} = \frac{k}{l} ; \tag{25.2}$$

this formula determines the critical value of a statically acting force P (for example, weight of mass *1*).

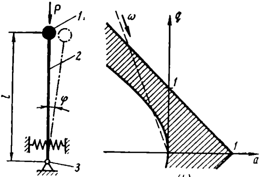
The same value can be found by considering free vibrations of mass *1*. In distinction to the equation of statics (25.1), the moment equation with respect to hinge *3* contains an inertia term and is of the form

$$Pl\varphi - k\varphi = ml^2\ddot{\varphi}, \tag{25.3}$$

i.e.,

$$\ddot{\varphi} + \frac{k-Pl}{ml^2}\varphi = 0.$$

When $k = Pl$, the frequency of free vibration of the system is zero, i.e., the system becomes unstable. The value of the critical force is given by the previous formula (25.2).



(a) (b)

Fig. 145

Let us now consider the case when the force P varies in time according to a harmonic law

$$P = P_0 + P_1 \cos \omega t.$$

The equation of vibration of the rod (25.3) is then written as

$$(P_0 + P_1 \cos \omega t)\, l\varphi - k\varphi = ml^2\ddot{\varphi},$$

.e.,

$$\ddot{\varphi} + \frac{1}{ml^2}(k - P_0 l - P_1 l \cos \omega t)\,\varphi = 0;$$

this equation leads to the standard form of Mathieu's equation (23.5) if we put

$$\left.\begin{aligned} 2\tau &= \omega t, \\ a &= \frac{4}{m\omega^2 l}\left(\frac{k}{l} - P_0\right), \\ q &= \frac{2P_1}{m\omega^2 l}. \end{aligned}\right\} \tag{25.4}$$

As the frequency ω increases, the parameters a and q decrease in proportion. A dashed ray in Fig. 143 indicates that the system passes through a succession of alternating stable and unstable states. The slope of the ray is defined by the ratio

$$k = \frac{q}{a} = \frac{P_1}{2(P_{cr} - P_0)},$$

where P_{cr} is the static critical force given by expression (25.2).

At a given value of P_1 the quantity k depends on the difference $P_{cr} - P_0$. The nearer the value of the static component P_0 approaches the critical value P_{cr}, the steeper is the ray and the wider are the areas of instability regions crossed by it. This is, of course, natural because the closeness of the force P_0 to the Euler force must facilitate the occurrence of instability.

It is important to note that the loss of stability is possible for arbitrarily small values of the compressive static component P_0 and even when its sign is reversed (i.e., when the static component is tensile). As is seen from Fig. 143, when $P_0 < 0$ the ray $q = ka$ has a small slope but it also crosses a number of iustability regions.

On the other hand, the Ince-Strutt diagram indicates that the stability of the system is possible for $P_0 = P_{cr}$ and even for $P_0 > P_{cr}$. Indeed, if $P_0 = P_{cr}$, then $a = 0$, the ray $q = ka$ coincides with the axis of ordinates of the Ince-Strutt diagram, but the system remains stable if $|q| < 1$. According to conditions (25.4), this requires the fulfilment of the inequality

$$\omega > \sqrt{\frac{2P_1}{ml}}.$$

When $P_0 > P_{cr}$, the ray $q = ka$ is located in the second quadrant of the Ince-Strutt diagram; Figure 145*b* shows that in this case, too, the stability of a system is possible in a suitably chosen range of frequency ω. Thus, under certain conditions the vibratory component of the compressive force can stabilize a system which is unstable without vibration.

26. Pendulum with a Vibrating Point of Suspension

Consider the pendulum represented in Fig. 146*a*. If the point of suspension is fixed, the only force producing a moment about the point of suspension is the weight of the mass mg; accordingly the equation of small oscillations of the pendulum is

$$-mgl\varphi = ml^2\ddot{\varphi}.$$

If, however, the point of suspension vibrates along the y axis according to the law

$$y = A\cos\omega t,$$

it is necessary, in setting up the moment equation, to take into account the inertia force due to the base motion $-m\ddot{y} = mA\omega^2\cos\omega t$;

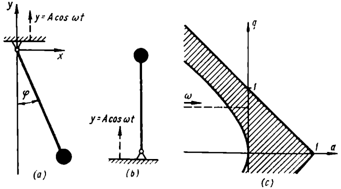


Fig. 146

its moment is $mA\omega^2 l\varphi\cos\omega t$ and the equation of oscillation of the pendulum is written as

$$-mgl\varphi + mA\omega^2 l\varphi\cos\omega t = ml^2\ddot{\varphi},$$

i.e.,

$$\ddot{\varphi} + \left(\frac{g}{l} - \frac{A\omega^2}{l}\cos\omega t\right)\varphi = 0; \tag{26.1}$$

this equation can be reduced to the standard form (23.5) if we put

$$2\tau = \omega t, \quad a = \frac{4g}{\omega^2 l}, \quad q = \frac{2A}{l}.$$

It is now immediately seen from the Ince-Strutt diagram that the parameter a is independent of the amplitude of vibration of the point of suspension, and no matter how small the amplitude A may be, the instability of the lower position of the pendulum occurs near the values $a = 1, 4, 9, \ldots$, i.e., when

$$\omega = 2\sqrt{\frac{g}{l}},\ \sqrt{\frac{g}{l}},\ \frac{2}{3}\sqrt{\frac{g}{l}}.$$

Let us now discuss the question of *stability of the upper position of the pendulum* (Fig. 146*b*). When the support is fixed, this position is of course unstable; however, the vibration of the foundation may render this position stable. To obtain the equation of motion for this case, it is sufficient to change the sign before the term containing the acceleration g in Eq. (26.1); accordingly the parameter a becomes negative

$$a = -\frac{4g}{\omega^2 l} \tag{26.2}$$

(the other quantities remain unaltered).

From Fig. 146*c* it is seen that the upper position of the pendulum may be stable. For small amplitudes A of vibration of the point of suspension (when $|q| < 1$) the stability of the upper position is provided if the inequality $|a| < q^2/2$ is satisfied. According to expression (26.2), this stability condition takes the form

$$\omega > \frac{\sqrt{2gl}}{A}.$$

27. Cases of Periodic Variation of the Inertia of a System

Let us solve an auxiliary problem of torsional vibrations of the system shown in Fig. 147*a*. Assume that only disk *1* and weight *4* possess mass in this system, the mass moment of inertia of disk *1* being J and the mass of the weight m. Shaft *2* is fixed at one end and can only twist. Rod *3* is perfectly rigid and serves only to connect disk *1* and weight *4* which can move horizontally; the length of the rod is considerably greater that the radius of the disk. This system has one degree of freedom.

Suppose that the disk executes free vibrations about the equilibrium position; it is clear that this process will involve weight *4* the effect of whose inertia can be determined by the energy method.

Let φ be the angle of rotation of the disk during vibration so that the law of free vibration of the disk is

$$\varphi = a \sin pt,$$

where p = natural frequency of system,
a = amplitude of torsional vibration of disk (Fig. 147*b*).

The maximum angular velocity of the disk is

$$\dot{\varphi}_{\max} = ap$$

and the maximum kinetic energy of the disk is given by

$$T_1 = \frac{J(\dot{\varphi}_{\max})^2}{2} = \frac{J(ap)^2}{2}.$$

From Fig. 147*b* it is seen that to the angle of rotation of the disk φ corresponds a horizontal displacement of point M equal to $r\varphi \sin \alpha$.

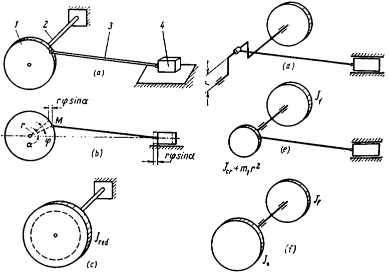


Fig. 147

If the length of rod *3* is large, it is permissible to assume that the displacement of weight *4* is of the same magnitude

$$x = r\varphi \sin \alpha.$$

The velocity of weight *4* is

$$\dot{x} = r\dot{\varphi} \sin \alpha.$$

The maximum value of the velocity of the weight is

$$\dot{x}_{\max} = rap \sin \alpha$$

so that the maximum kinetic energy of the weight is equal to

$$T_2=\frac{m\dot{x}^2_{\max}}{2}=\frac{m\,(rap\sin\alpha)^2}{2}.$$

Consequently, the maximum kinetic energy of the system is

$$T=T_1+T_2=\frac{(ap)^2}{2}\left[J+\frac{mr^2}{2}(1-\cos 2\alpha)\right].$$

Hence, the quantity

$$J_*=J+\frac{mr^2}{2}(1-\cos 2\alpha) \qquad (27.1)$$

may be interpreted as the *reduced mass moment of inertia* of the disk, which makes it possible to consider an equivalent system (Fig. 147*c*) instead of the given one (Fig. 147*a*).

From expression (27.1) it is seen that the effect of the inertia of weight 4 on the process of vibration depends on the angle α.

Based on expression (27.1), we proceed to a vibration problem for a simple slider-crank mechanism (Fig. 147*d*).

If the mass of the connecting rod is replaced by two masses one of which (m_1) rotates with the crank and the other moves with the piston, we obtain the scheme shown in Fig. 147*e*. Consequently, to the moment of inertia of the crank J_{cr} must be added the moment of inertia of the additional effective mass of the connecting rod m_1r^2.

On the basis of the foregoing solution of the anxiliary problem we can proceed from this scheme to a scheme (Fig. 147*f*) consisting of two disks: one of these (flywheel) has a moment of inertia J_f and the other has a moment of inertia given by formula (27.1) as

$$J_*=J_{cr}+m_1r^2+\frac{m_2r^2}{2}(1-\cos 2\omega t); \qquad (27.2)$$

here m_2 is the combined mass of the piston and the corresponding portion of the connecting rod.

The equation of vibration of this system is of the form [see formulas (4.8) and (4.11)]

$$\ddot{\varphi}+\frac{k\,(J_*+J_f)}{J_*J_f}\varphi=0.$$

Substituting expression (27.2), we obtain

$$\ddot{\varphi}+k\left[\frac{1}{J_f}+\frac{1}{J_*\left(1-\frac{m_2r^2}{2J_0}\cos 2\omega t\right)}\right]\varphi=0, \qquad (27.3)$$

where

$$J_0=J_{cr}+m_1r^2+\frac{m_2r^2}{2}$$

is the *average value of the reduced moment of inertia* of the crank.

From Eq. (27.3) it is seen that the coefficient of the function φ depends on the time t as in the cases considered above. However, in this problem the variability of the coefficient results from the periodic variation of the *moment of inertia* and not the stiffness.

Since the fraction $m_2r^2/2J_0$ is small in comparison with unity, it is permissible to assume

$$\frac{1}{1-\frac{m_2r^2}{2J_0}\cos 2\omega t} \simeq 1+\frac{m_2r^2}{2J_0}\cos 2\omega t.$$

Equation (27.3) then becomes

$$\ddot{\varphi}+k\left(\frac{1}{J_f}+\frac{1}{J_0}\right)\left(1-\frac{J_f m_2 r^2}{2J_0(J_f+J_0)}\cos 2\omega t\right)\varphi=0. \qquad (27.4)$$

To determine regions of parametric resonance we represent this expression in standard form (23.5), putting

$$\omega t=\tau, \quad a=\frac{k(J_0+J_f)}{J_0J_f\omega^2}, \quad q=\frac{km_2r^2}{4J_0^2\omega^2}.$$

Calculating the values of these parameters, we can check the stability of the system using the Ince-Strutt diagram.

CHAPTER VI

SELF-EXCITED FRICTIONAL VIBRATIONS

28. Nature of Self-Excited Vibrations

In all the foregoing systems with inelastic resistances, steady-state vibratory processes were possible only so far as there were external effects of a periodic type. However, in some systems steady-state vibrations may be set up *without any periodic effects from outside*; these systems are called *self-excited systems.* Since some energy is inevitably dissipated in all real vibratory processes, every

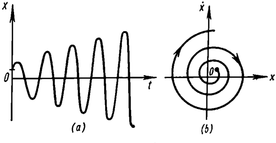


Fig. 148

self-excited system has a source of energy which makes up for the energy losses but, by its nature, possesses no vibratory properties.

Self-excitation of vibrations often depends upon the fact that the state of equilibrium of a system is unstable; therefore, when this state is disturbed in any possible way the resulting vibration will not be of a damped type but, on the contrary, will continually build up; the forces produced by the source of energy and contributing to the swinging of the system depend themselves on its motion (in the equilibrium position they are equal to zero). The process of self-excitation is illustrated in Fig. 148*a*; the corresponding phase trajectory is shown in Fig. 148*b*; in this case the point representing the equilibrium position is called an *unstable focus.*

Of special interest are self-excited mechanical vibrations occurring in systems with friction. The friction force, which was the cause of damping of vibrations in the examples previously considered, may be responsible for their building-up.

A clue to the reason for this may be obtained by considering a simple system (Fig. 149*a*). The system consists of two rotating drums driving an infinite belt; the belt carries a weight of mass m whose motion is restrained by a spring of stiffness k. The friction force developed when the weight is sliding tends to move the weight to the right and produces an extension of the spring. Let the friction force in the equilibrium position of the weight be R_0; the static displacement of the weight is then

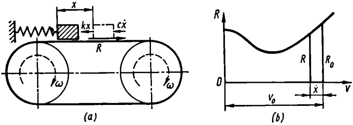


Fig. 149

$$x_0 = \frac{R_0}{k}. \tag{28.1}$$

For further discussion it is necessary to take into account that the friction force R depends on the relative velocity v (at a given normal pressure of the weight which will be assumed constant). Let the friction characteristic be of the form shown in Fig. 149*b*; the values v_0 and R_0 correspond to the state of equilibrium of the weight.

Suppose that the weight is displaced from the position of rest at time $t = 0$ due to some disturbance; let us see what the ensuing motion will be assuming that the velocity of the belt is constant during the vibration of the weight. Let the displacement of the weight at the current instant t be x, its velocity $\dot{x}$ and the velocity of sliding $v = v_0 - \dot{x}$.

At this instant the weight is acted on by three forces: the reaction of the spring $-kx$, the friction force R and the external resistance which will be assumed proportional to the velocity and expressed as $-c\dot{x}$.

The force R differs from the force R_0 since the relative velocity v differs from the velocity v_0. For small fluctuations of the relative velocity it may be taken that

$$R = R_0 - R_0'\dot{x}, \tag{28.2}$$

where R_0' is the slope of the friction characteristic at the point with co-ordinates $v_0 R_0$.

The equation of motion of the weight is

$$-kx - c\dot{x} + R = m\ddot{x}.$$

Substituting expression (28.2), we obtain

$$m\ddot{x} + kx - R_0 + (R_0' + c)\dot{x} = 0. \tag{28.3}$$

We shift the origin of displacements to the point $x = x_0$, i.e., we introduce a new variable $x_1 = x - x_0$.

Equation (28.3) then becomes

$$m\ddot{x}_1 + (R_0' + c)\dot{x}_1 + kx_1 + kx_0 - R_0 = 0.$$

According to expression (28.1), the sum of the last two terms is zero, therefore we obtain the equation

$$m\ddot{x}_1 + (R_0' + c)\dot{x}_1 + kx_1 = 0. \tag{28.4}$$

An inspection of the structure of this equation indicates that, after a certain time, the vibrations must disappear if the sum $R_0' + c$ is positive. This is always the case for the ascending portion of the friction characteristic where $R_0' > 0$.

However, for small values of v_0 (descending portion of the friction characteristic) the quantity R_0' becomes negative (Fig. 149b). If $c > |R_0'|$, the sum $R_0' + c$ remains positive and the vibration will be a damped one. If the sum $R_0' + c$ is zero (i.e., $c = |R_0'|$), the damping term in Eq. (28.4) vanishes and *vibrations with constant amplitude* become possible. If, however, the sum $R_0' + c$ is negative, the system possesses *"negative damping"* and the vibrations will *build up* in the course of time due to the energy transmitted by the moving belt.

The nature of "negative damping" for a descending characteristic can be understood from the following argument. When the weight moves to the right, i.e., in the direction of motion of the belt, the relative velocity of sliding decreases; at the same time the friction force increases and its increment is directed to the right, i.e., *it has the same direction as the motion*. In the other interval of motion when the weight moves to the left, the increment of the friction force is also directed to the left, i.e., *it has again the same direction as the* *motion*. This type of variation of the friction force is the cause of the intensification of vibrations.

Thus, for vibrations to build up, the condition $R_0' + c < 0$ must be fulfilled, which is possible only when the descending characreristic is sufficiently steep. The above condition is usually satisfied only for small velocities v_0.

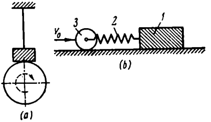


Fig. 150

Following the same line of reasoning we may conclude that self-excited vibrations are possible for an elastically restrained block pressed against a rotating disk (Fig. 150*a*) and also for a weight attached to a spring (Fig. 150*b*) when its left end is given a motion with constant velocity; in these cases, too, the necessary condition for self-excited vibrations is the existence of a descending portion of the friction characteristic.

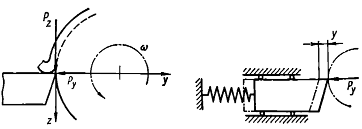


Fig. 151

Self-excited vibrations arising during metal cutting on machines fall in this category of phenomena. Without going into details, we consider the nature of self-excited vibrations occurring during cutting on a turning lathe. The workpiece exerts on the cutting tool a reaction P, which can be resolved into components P_y and P_z (Fig. 151). The tool is elastically restrained and its end may execute vibrations both in the horizontal and vertical direction. To see whether self-excited vibrations are possible, it is sufficient to consider vibrations only in the horizontal direction $y = y(t)$ and to take into account an important experimental result, viz. that the horizontal component P_y depends on the velocity of horizontal vibrations of the tool $\dot{y}$.

Thus, the tool is acted on by three forces: the elastic force $-ky$ of the tool-support system (k is the spring constant), the reaction

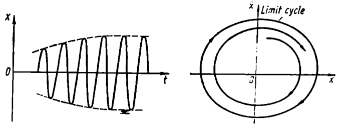


Fig. 152

of the workpiece $-P_y(\dot{y})$ and the sum of various inelastic resistances which may be combined into one term of the form $-c\dot{y}$.

Thus, the equation of motion of the tool-support system is

$$-P_y(\dot{y}) - ky - c\dot{y} = m\ddot{y}$$

(m is the reduced mass of the tool-support system). After linearizing the force P_y according to expression (28.2) we arrive again at an equation of the type of (28.4). Consequently, in this case, too, self-excited vibrations are possible if the force P_y has a descending characteristic.

As ascertained, the instability of the state of equilibrium can be revealed on the assumption of small vibrations, i.e., by means of a linear approximation. However, if we discard this assumption and trace the further development of the process, it will be found that the growth of amplitudes falls off gradually and *the amplitude eventually settles down to a practically constant value.* This process of steady (stationary) self-excited vibration is shown in Fig. 152 and is called the *limit cycle*. An important feature of the limit cycle is that it is wholly *independent of the initial conditions*; when the state of equilibrium is disturbed in any possible way the system approaches one and the same limit cycle. The determination of the parameters (frequency, amplitude) of steady self-excited vibration calls for an analysis of the corresponding non-linear problem.

In some cases steady self-excited vibrations are almost harmonic and have a frequency equal to the frequency of free vibration of the system; the corresponding systems are called *quasi-linear systems*. In other cases steady self-excited vibrations differ sharply from harmonic vibrations, they are accompanied by stops and jumps in velocity; such vibrations (and the corresponding systems) are called *relaxation* or *discontinuous* vibrations (systems).

29. Self-Excited Vibrations of Quasi-Linear Systems

Basic Equation

In the above discussion we assumed a linear approximation for the friction force developed during vibration. This sufficed for the investigation of the stability of the state of equilibrium, but for the analysis of the *settling-down process* and also the *process of steady self-excited vibration* it is necessary to take into account non-linear terms.

We write the expansion

$$R = R_0 - R_0'\dot{x}_1 + \frac{1}{2} R_0''\dot{x}_1^2 - \frac{1}{6} R_0'''\dot{x}_1^3 \qquad (29.1)$$

and retain the above-mentioned terms*. The equation of motion then changes from (28.4) to

$$m\ddot{x}_1 + kx_1 + (R_0' + c)\dot{x}_1 - \frac{R_0''}{2}\dot{x}_1^2 + \frac{R_0'''}{6}\dot{x}_1^3 = 0. \qquad (29.2)$$

It is here assumed that the velocity v_0 corresponds to the descending portion of the characteristic.

Steady Self-Excited Vibrations

To solve Eq. (29.2) we use the *energy balance method*. Suppose that the steady self-excited vibrations may be approximated by a harmonic law

$$x_1 = a \sin pt \qquad (29.3)$$

with the frequency p of free vibration of the system. Expression (29.3) becomes perfectly accurate if the variable portion of the friction force R expressed by the last three terms of Eq. (29.1)

$$\tilde{R} = (R_0' + c)\dot{x}_1 - \frac{R_0''}{2}\dot{x}_1^2 + \frac{R_0'''}{6}\dot{x}_1^3 \qquad (29.4)$$

---

* Expression (29.1) follows from the expansion

$$R = R_0 + \left(\frac{dR}{dv}\right)_0 \Delta v + \frac{1}{2}\left(\frac{d^2R}{dv^2}\right)_0 (\Delta v)^2 + \frac{1}{6}\left(\frac{d^3R}{dv^3}\right)(\Delta v)^3 + \dots$$

after substituting $\Delta v = -\dot{x}_1$.

is identically zero. Of course, calculating the sum of these terms with the use of expression (29.3), we see that this identity is not fulfilled; however, to obtain an approximate solution we may restrict ourselves to a less severe requirement, viz. that the work done by the friction forces per cycle be zero. Although this solution will not ensure a constant energy of the system for any instant, it corresponds to a constant average energy per cycle.

The work done by the friction force $\tilde{R}$ during the time dt is

$$\tilde{R}\,dx_1 = \tilde{R}\,\frac{dx_1}{dt}\,dt = \tilde{R}\dot{x}_1\,dt;$$

consequently, the work done by the friction force during the period T is

$$\int_0^T \tilde{R}\dot{x}_1\,dt = 0,$$

where T is the period of self-excited vibration assumed to be equal to the natural period.

Substituting the friction force $\tilde{R}$ from expression (29.4), we obtain

$$\int_0^T \left[(R_0' + c)\,\dot{x}_1^2 - \frac{R_0''}{2}\,\dot{x}_1^3 + \frac{R_0'''}{6}\,\dot{x}_1^4\right]dt = 0. \tag{29.5}$$

According to expression (29.3)

$$\dot{x}_1 = ap\cos pt,$$

consequently,

$$\int_0^T \left[(R_0' + c)\cos^2 pt - \frac{R_0''}{2}\,ap\cos^3 pt + \frac{R_0'''}{6}\,a^2p^2\cos^4 pt\right]dt = 0.$$

After integration we find the amplitude of self-excited vibration

$$a = \frac{2}{p}\sqrt{-\frac{2\,(R_0' + c)}{R_0'''}}\,. \tag{29.6}$$

Since the sum $R_0' + c$ is negative, the radicand is positive only if the third derivative R_0''' is positive.

It is usually possible to neglect all inelastic resistances except the friction force R. Putting $c = 0$, we obtain a simpler formula for the amplitude

$$a = \frac{2}{p}\sqrt{-\frac{2R_0'}{R_0'''}}\,. \tag{29.7}$$

Example 30. Determine the amplitude of self-excited vibration for the case when the friction characteristic is given by the equation (Fig. 153)

$$R = 3R_*\left(1 - \frac{v}{v_*} + \frac{v^3}{3v_*^2}\right), \tag{29.8}$$

where R_* and v_* are, respectively, the friction force and the relative velocity at the point where the characteristic has a minimum value.

The rated velocity of sliding v_0 is in the descending portion of the characteristic near the minimum value of v and is equal to $0.95v_*$.

Differentiating the given equation we find

$$R' = \frac{3R_*}{v_*}\left(\frac{v^2}{v_*^2} - 1\right), \quad R'' = \frac{6R_*}{v_*^2}.$$

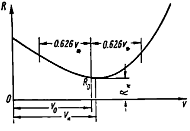


Fig. 153

Substituting $v_0 = 0.95v_*$ in the expression for R', we obtain

$$R_0' = \frac{0.2925R_*}{v_*}.$$

By formula (29.7) we find the amplitude of self-excited vibration

$$a = \frac{2v_*}{p}\sqrt{0.0975} = \frac{0.626v_*}{p}.$$

The maximum velocity of self-excited vibration is

$$\dot{x}_{1\,\max} = ap = 0.626v_*.$$

Figure 153 shows the range of values of the velocity $\dot{x}_1$; the fluctuations are very great in comparison with v_0.

The maximum velocities of vibration for other ratios v_0/v_* have the following values:

v_0/v_*	0.900	0.925	0.950	0.975	1.000
$v_{1\max}/v_*$ . . .	0.873	0.761	0.626	0.445	0.000

For $v_0 > v_*$ self-excited vibration is impossible since R_0' is positive. For $v_0 < 0.9v_*$ no calculations were made—the velocity fluctuations become so significant that the relative velocity falls to zero and in certain time intervals the weight has no motion with respect to the belt (i.e., it moves with uniform velocity). This makes it impermissible to use the harmonic law of motion (29.3).

Settling-Down Process

The energy balance method may be used not only to determine the amplitude of steady self-excited vibration but also to investigate the settling-down process. To do this, it is necessary to proceed not from expression (29.3), which corresponds to the motion with constant amplitude, but from a more general expression

$$x_1 = a(t)\sin pt$$

assuming that *the amplitude gradually varies.*

However, this variation may be considered slow if the increment Δa per cycle is small in comparison with the amplitude a. In this case the expression for the velocity is approximated by

$$\dot{x}_1 = a(t)\,p\cos pt.$$

Accordingly the work done by the friction force per cycle is expressed by integral (29.5). In the present case *this integral is not zero* but represents the increment of the energy of the system per cycle

$$\int_0^T \left[(R_0' + c)\,\dot{x}_1^2 - \frac{R_0''}{2}\,\dot{x}_1^3 + \frac{R'''}{6}\,\dot{x}_1^4 \right] dt = \Delta U \tag{29.9}$$

or, evaluating (on the assumption of constant amplitude a), we find

$$\pi p a^2 \left(R_0' + c + \frac{1}{8} R_0''' a^2 p^2 \right) = \Delta U.$$

The increment ΔU can also be calculated by comparing the potential energy at two consecutive instants when the system has a maximum deflection

$$U = \frac{ka^2}{2},$$

$$U + \Delta U = \frac{k(a+\Delta a)^2}{2}.$$

By subtracting, we obtain

$$\Delta U \cong ka\Delta a. \tag{29.10}$$

Equating expressions (29.10) and (29.9), we find

$$\pi p a^2 \left(R_0' + c + \frac{1}{8} R_0''' a^2 p^2 \right) = ka\Delta a. \tag{29.11}$$

Regarding now a as a continuous function of the argument t, we replace

$$\Delta a = \frac{da}{dt} T.$$

Instead of expression (29.11) we obtain the differential equation

$$\frac{da}{dt} = \frac{a}{2m} \left(R_0' + c + \frac{1}{8} R_0''' a^2 p^2 \right).$$

Integrating this differential equation with the initial condition $a = a_0$ at $t = 0$, we obtain the equation of the envelope (*settling-down equation*)

$$a = \frac{b}{\sqrt{\left(1 + \frac{b^2}{a_0^2}\right) e^{-\frac{c+R_0'}{m} t} - 1}}, \tag{29.12}$$

where

$$b^2 = \frac{8(R_0' + c)}{p^2 R_0'''}.$$

As $t \to \infty$ the equation reduces to formula (29.6) for the amplitude of steady self-excited vibration.

The settling-down process and also the limit cycle can be obtained in the phase plane by a semigraphical procedure using the *delta method*. To do this, we denote $x_1 = y$

$$\frac{dy}{dt} = v. \tag{29.13}$$

The differential equation (29.2) then becomes

$$\frac{dv}{dt} = f(v) - p^2 y, \tag{29.14}$$

where

$$f(v) = -\frac{R_0' + c}{m} v + \frac{R_0''}{2m} v^2 - \frac{R_0'''}{6m} v^3. \tag{29.15}$$

Dividing Eq. (29.14) by (29.13), we obtain the basic *differential equation of phase trajectories*

$$\frac{dv}{dy} = \frac{f(v) - p^2 y}{v}. \tag{29.16}$$

The set of integral curves of this differential equation forms the phase diagram of the system.

We introduce the *dimensionless time* $\tau = pt$ and denote

$$\nu = \frac{dy}{d\tau} = \frac{v}{p}.$$

The differential equation (29.16) then becomes

$$\frac{d\nu}{dy} = -\frac{\delta(\nu) + y}{\nu}, \tag{29.17}$$

where

$$\delta(\nu) = -\frac{f(p\nu)}{p^2}. \tag{29.18}$$

For short intervals of time and accordingly small increments of ν the quantity δ may be considered to be constant. The variables y and ν in the differential equation (29.17) are then separable and, after integration, we obtain the *algebraic equation*

$$\nu^2 + (y + \delta)^2 = \text{constant};$$

this equation describes a circle whose centre is located on the axis of abscissas at the point $y = -\delta$, $\nu = 0$. Thus, for a short interval of time the segment of the phase trajectory represents a circular arc with centre at that point.

The construction of the phase trajectory begins with a point having the co-ordinates $y_0 = y(0)$, $\nu_0 = \nu(0)$ defined by the initial conditions at $\tau = 0$. The value ν_0 is substituted in expression (29.18) and $\delta(\nu_0)$ is found. This value of δ determines the abscissa of the centre of the circle (Fig. 154).

It is now possible to draw a small circular arc from the starting point of the phase trajectory (y_0, v_0) in a clockwise direction; in this way the first element of the phase trajectory is determined. A new value of the phase co-ordinate v_1 can be taken from the drawing and substituted in expression (29.18) to find $\delta(v_1)$, i.e., to determine the position of a new centre of the circle on the axis of abscissas. Using this centre, the second element of the phase trajectory is constructed, and so on.

Of course, in performing the construction it is necessary to follow the general rules of graphical solutions and, particularly, to avoid taking too large lengths of successive segments.

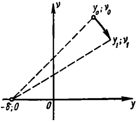


Fig. 154

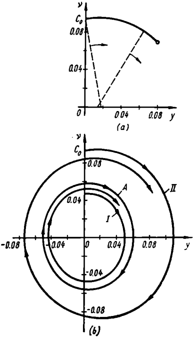


Fig. 155

Example 31. Construct a phase curve for the process of settling down of self-excited frictional vibrations defined by the differential equation (29.2) if the friction force is given by (29.8). The parameters of the system are: $m = 0.102$ kgf-sec/cm, $k = 1{,}000$ kgf/cm, $R_* = 10$ kgf, $v_0 = 9.5$ cm/sec, $v_* = 10$ cm/sec.

From expression (29.8) we find

$$R_0' = \frac{3R_*}{v_*}\left(\frac{v_0^2}{v_*^2} - 1\right) = -0.2925 \text{ kgf-sec/cm},$$

$$R_0'' = \frac{6R_* v_0}{v_*^3} = 0.57 \text{ kgf-sec}^2/\text{cm}^2, \qquad R_0''' = \frac{6R_*}{v_0^3} = 0.06 \text{ kgf-sec}^3/\text{cm}^3$$

and expression (29.15) becomes

$$f(v_1) = +2.87v + 2.80v^2 - 0.10v^3.$$

Determine the frequency of self-excited vibration

$$p = \sqrt{\frac{k}{m}} = 100 \text{ sec}^{-1}.$$

From expression (29.18) we find

$$\delta(\nu) = -0.029\nu - 2.80\nu^2 + 10\nu^3 \tag{29.19}$$

and proceed to construct the phase trajectory starting with any point (specified by the initial conditions) in the phase plane. Let $y_0 = 0$, $\nu_0 = 0.09$. Then $\delta = -0.018$ by formula (29.19).

Figure 155*a* shows the beginning of the construction. The first segment of the phase trajectory has a centre at a point with co-ordinates 0.018, O. From this centre a circular arc is drawn from the starting point C_0 to a point C_1 at which $\nu = 0.08$. Formula (29.19) gives $\delta = -0.015$. The centre of the second arc is at a point with co-ordinates 0.015, O. From this centre the second arc is drawn to a point C_2 at which $\nu = 0.07$, and so on.

The complete phase trajectory is shown in Fig. 155*b* and marked *II*. It represents a *converging spiral*. Another phase trajectory starting at point O, 0.045 is a *diverging spiral*; it is marked *I*. Phase trajectories of the type *I* and *II* approach indefinitely a closed trajectory *A*, which is the limit cycle.

Curve *A* is asymmetric, the asymmetry being most pronounced about the vertical axis. The maximum and minimum displacements of the system when it moves on the limit cycle are 0.06 and 0.05 cm, respectively. Thus, the centre of oscillation is somewhat shifted in the direction of the positive y axis and one-half of the total excursion is 0.055 cm. The maximum value of ν is 0.055 cm and the maximum velocity is $v_{\max} = \nu p = 100 \times 0.055 = 5.5$ cm/sec. These results are in fair agreement with solution (29.6) according to which the amplitude of self-excited vibration is $a = 0.064$ cm and the maximum velocity is $v_{\max} = ap = 6.4$ cm/sec.

In this case the results of the semigraphical solution by the delta method should be considered more accurate; in any case it is free from the arbitrary assumption of the harmonic nature of the process which was made in the analytical solution by the energy method.

30. Self-Excited Relaxation Vibrations

Self-Excited Vibrations of a Massless System

The foregoing method of determining the amplitude of self-excited vibration can be considered satisfactory only in cases of weak non-linearity when the friction force is small and the vibrations may be assumed sinusoidal.

Below is given an approximate method which applies to the case of a *very large friction force*. We revert to the scheme of Fig. 149 and assume that the motion is determined by this force and the elastic force only and that the inertia force of the weight may be neglected. The equation of motion of this *massless degenerate system* is of a purely static form

$$kx = R(v), \tag{30.1}$$

where kx = spring force,

$R(v)$ = friction force, which is a function of relative velocity.

It should be remembered that relation (30.1) establishes the proportionality between the co-ordinate x and the friction force R; since the co-ordinate x cannot vary abruptly in time, the friction force must also be a *continuous function* of the time t.

In Fig. 156, point O corresponds to the state of equilibrium of the weight, the friction force R_0 balancing the elastic force kx_0 (x_0 is the static deflection of the weight). Check that this state of equilibrium is unstable.

Let the weight be pulled out an additional distance Δx by a force ΔP and held in this position; since the weight is stationary, the relative velocity is v_0 and the state of the system is characterized by the same point O.

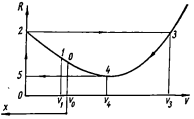


Fig. 156

Let the force ΔP be released and the weight left alone at a certain instant which will be taken as the initial one. Since the initial elastic force $k(x_0 + \Delta x) = R_0 + \Delta P$ is larger than R_0, equality (30.1) requires the existence of a friction force R_1 larger than R_0. As can be seen from Fig. 156, an instantaneous increase of the friction force from R_0 to R_1 can result only from a *sudden change* of the relative velocity, viz. a decrease to a value v_1. To the beginning of vibration corresponds point *1*. Of course, an *instantaneous* change of the velocity is only a convenient approximation of a *very rapid* change of the velocity.

The decrease of the relative velocity to the value v_1 brings about a positive absolute velocity of the weight $\dot{x} = v_0 - v_1$. Consequently, the moment the weight pulled out to the right is left alone, it immediately acquires the velocity $\dot{x}$ also directed to the right. This causes the weight to move further to the right away from the position of equilibrium. To this process corresponds the portion of the characteristic *1-2*; from the kinematic point of view it is characterized by a continuous growth of both the co-ordinate x and the velocity $\dot{x}$, while from the static point of view, by a simultaneous (and equal) increase of both the spring force and the friction force. At point *2* the relative velocity vanishes, i.e., the absolute velocity $\dot{x}$ becomes equal to the velocity of the belt v_0 and the weight ceases to slide on the belt.

A continuation of the motion to the right would mean increasing the elastic force kx and formula (30.1) would require, accordingly, a larger friction force R.

However, the friction force can no longer increase; hence, as soon as the generic point reaches position *2*, *the motion of the weight to the right ceases instantaneously.*

To determine the new value of the velocity $\dot{x}$, it should be borne in mind that the friction force R must remain unchanged. Consequently, the new value of the velocity corresponds to the generic point *3*.

Since $v_3 > v_0$, the velocity of the weight $\dot{x}$ changes sign. Consequently, after an instantaneous change of the velocity from $\dot{x}_2$ to $\dot{x}_3$ the weight begins to move to the left. During this motion, the deformation of the spring will diminish and so will the elastic force and the friction force R. A decrease in the force R can result only from a reduction in the relative velocity v and corresponds to portion *3-4*; in this portion the relative velocity will decrease continuously up to point *4*.

However, further motion cannot be described by the branch 4-0-1-2 as this branch corresponds to increasing friction force R. This is inconsistent with Eq. (30.1), which indicates that, when the weight moves to the left, the elastic force must decrease and not increase.

Therefore, at point *4* the velocity must again undergo an instantaneous change—the generic point jumps to position *5*. As in the portion *2-3*, the jump *4-5* means a discontinuity in the velocity with a constant value of R. Then the weight will move without sliding to the right together with the belt and the elastic force will again begin increasing; the friction force R will also increase. This increase corresponds to portion *5-2*.

At point *2*, the velocity will again undergo a discontinuity, the generic point will move to position *3*, and so on. In this manner a periodic motion occurs on the cycle *2-3-4-5-2-3*... without returning to the branch 4-0-1-2.

The foregoing vibratory cycle is set up for any initial disturbance if the starting generic point lies somewhere on the descending branch of the friction characteristic, i.e., for a sufficiently low operating velocity v_0.

The complete cycle of self-excited vibration consists of two stages (starting from point *5*).

First stage. The motion of the weight at a constant velocity v_0 to the right together with the belt (portion *5-2*). The displacement x is determined in terms of the friction force from formula (30.1) and has the following values: $x_5 = R_5/k$ at the beginning of the stage (point *5*) and $x_2 = R_2/k$ at the end of the stage (point *2*).

Second stage. The motion of the weight to the left with decreasing velocity (portion *3-4*). At the beginning of the stage (point *3*) the

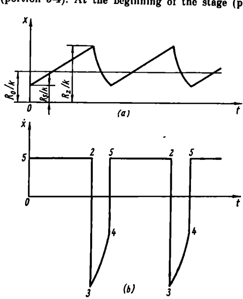


Fig. 157

velocity is $\dot{x}_3 = -(v_3 - v_0)$ and the displacement is $x_3 = R_3/k$ with $x_3 = x_2$. At the end of the stage (point *4*) the velocity is $\dot{x}_4 = -(v_4 - v_0)$ and the displacement is $x_4 = R_4/k$ $(x_4 = x_5)$.

Figure 157*a* and *b* shows the graph of the motion $x = x(t)$ and the graph of the velocity $\dot{x} = \dot{x}(t)$. The graph of the motion is of a saw-tooth shape and differs sharply from the law of harmonic motion. The maximum displacements from the value x_0 are unequal in the two directions; it is therefore more convenient to speak not of the amplitude but of the *total excursion* of self-excited vibration; the latter is defined as the sum

$$|x_2|+|x_5|=\frac{R_2-R_5}{k} \tag{30.2}$$

and can be calculated from the friction characteristic.

The duration of the first stage of the motion is determined using the laws of uniform motion

$$t_1=\frac{R_2-R_5}{kv_0}\,. \tag{30.3}$$

The calculation of the duration of the second stage of the motion is somewhat more complicated.

Consider the derivative

$$R'_{\dot{x}}=\frac{dR}{d\dot{x}}$$

taken with respect to the velocity $\dot{x}$. From Eq. (30.1) we obtain

$$dR=k\,dx=k\dot{x}\,dt.$$

Consequently,

$$\frac{dR}{d\dot{x}}=k\dot{x}\frac{dt}{d\dot{x}}\,;$$

from this we find the duration of the second stage of the motion

$$t_2=\frac{1}{k}\int_{\dot{x}_3}^{\dot{x}_4}\frac{R'_{\dot{x}}}{\dot{x}}\,d\dot{x}. \tag{30.4}$$

The lower limit of integration corresponds to the generic point *3*, where $\dot{x}_3=v_0-v_3$; the upper limit corresponds to point *4* for which $\dot{x}_4=v_0-v_4$.

The period of self-excited vibration is

$$T=t_1+t_2.$$

Example 32. As in Example 31, the friction characteristic is given by

$$R=3R_*\left(1-\frac{v}{v_*}+\frac{v^3}{3v_*^3}\right)=3R_*\left[1-\frac{v_0-\dot{x}}{v_*}+\frac{(v_0-\dot{x})^3}{3v_*^3}\right]$$

and $v_0=0.5v_*$. Find the period and the total excursion of self-excited vibration.

From the friction characteristic we find

$$v_3=\sqrt{3}v_*=3.468v_0,\qquad R_3=R_2=3R_*,$$

$$v_4=v_*=2v_0,\quad R_4=R_5=R_*.$$

The limits of integration in formula (30.4) are

$$\dot{x}_3 = v_0 - v_3 = -2.468 v_0, \quad \dot{x}_4 = v_0 - v_4 = -v_0.$$

The duration of the first stage of the motion is, from formula (30.3),

$$t_1 = \frac{3R_* - R_*}{kv_0} = \frac{2R_*}{kv_0}.$$

The derivative of R with respect to the velocity $\dot{x}$ is

$$R'_{\dot{x}} = \frac{3R_*}{v_*}\left(\frac{3}{4} + \frac{\dot{x}}{v_*} - \frac{\dot{x}^2}{v_*^2}\right).$$

By formula (30.4), the duration of the second stage of the motion

$$t_2 = \frac{3R_*}{kv_*}\int_{-2.468v_0}^{-v_0}\left(\frac{3}{4} + \frac{\dot{x}}{v_*} - \frac{\dot{x}}{v_*^2}\right)\frac{d\dot{x}}{\dot{x}} = \frac{1.023R_*}{kv_0}$$

is approximately one half that of the first stage.

The period of self-excited vibration is

$$T = t_1 + t_2 = \frac{3.02R_*}{kv_0}.$$

The total excursion of vibration is, from formula (30.2),

$$\frac{3R_* - R_*}{k} = \frac{2R_*}{k}.$$

Self-Excited Vibrations with a Simplified Friction Characteristic

A simultaneous consideration of the mass of a vibrating object and large friction forces presents difficulties in the general case. But in some cases this analysis becomes necessary. Thus, for example, a smooth motion with very low feeding velocities is not always realized in metal-cutting machines. Instead, an intermittent motion takes place. This is important for heavy machines, where low feeding velocities are used and the weights of moving units are very large. In such cases the process is essentially non-linear and at the same time the dimensions of vibrating objects admit no assumption of the system being "massless".

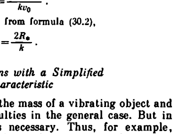


Fig. 158

An approximate solution of such problems can be carried out by using the simplified friction characteristic presented in Fig. 158. This characteristic gives two values for the friction force: the maximum static-friction force R_1 and the constant kinetic-friction force R_2.

Consider the motion of a heavy weight *1* (Fig. 150*b*) attached to a driving member *3* by spring *2*. The velocity of the driving member is assumed to be constant and equal to v_0.

Let weight *1* and member *3* move with the same velocity v_0. The spring force P is equal to the friction force R_2. If the velocity v_0 is very small, a minor casual obstacle may prove sufficient to stop the weight. Let us see what happens next.

The driving member, moving on to the right, will compress the spring until the compressive force P becomes equal to the maximum static-friction force R_1. Since no further growth of the friction force is possible, weight *1* "starts". The friction force R_1 then drops instantaneously to the value R_2, whereas the force $P = R_1$ of the compressed spring cannot change instantaneously and will remain equal to R_1 at the first instant of the ensuing motion. The moment of starting is taken as the origin of time ($t = 0$); both the displacement x and the velocity of the weight $\dot{x}$ are then equal to zero, i.e.,

$$x = 0, \qquad \dot{x} = 0^*. \tag{30.5}$$

Consider the subsequent motion of the weight. By the current instant the length of the spring will have changed by an amount $x - v_0 t$ and the spring force will have decreased to a value

$$P(t) = R_1 - k(x - v_0 t). \tag{30.6}$$

The differential equation of motion of the weight is written as

$$R_1 - k(x - v_0 t) - R_2 = m\ddot{x}$$

or

$$\ddot{x} + p^2 x = p^2 v_0 t + \frac{R_1 - R_2}{m}.$$

The solution of this equation corresponding to the initial conditions (30.5) is

$$x = v_0 t - \frac{v_0}{p}\sin pt + \frac{R_1 - R_2}{k}(1 - \cos pt). \tag{30.7}$$

The first term on the right-hand side expresses the motion with a velocity equal to that of the driving member and the other terms express additional vibrations of the weight.

---

[*] In this case one cannot assume an instantaneous change of the velocity. To such a change corresponds an infinite acceleration and hence an infinite force; here, the weight is acted on by a finite force $R_1 - R_2$ at the first instant of the motion.

The velocity of the weight varies according to the law

$$\dot{x} = v_0 - v_0 \cos pt + \frac{p(R_1 - R_2)}{k} \sin pt. \tag{30.8}$$

The weight comes to rest at the instant when $\dot{x}$ becomes zero again. According to relation (30.8), the condition for stopping $\dot{x} = 0$ leads to the equation

$$v_0 - v_0 \cos pt_1 + \frac{p(R_1 - R_2)}{k} \sin pt_1 = 0,$$

where t_1 is the time elapsed between the moment of starting and a new stop.

We introduce the dimensionless parameter

$$\alpha = \frac{p(R_1 - R_2)}{kv_0} = \frac{g}{pv_0} \Delta f, \tag{30.9}$$

where Δf is the difference between the static and kinetic coefficients of friction.

The condition for stopping becomes

$$\alpha \sin pt_1 = \cos pt_1 - 1. \tag{30.10}$$

Solving this transcendental equation, we obtain

$$\sin pt_1 = -\frac{2\alpha}{1+\alpha^2}, \quad \cos pt_1 = \frac{1-\alpha^2}{1+\alpha^2}. \tag{30.11}$$

After finding the value t_1, we can determine from formula (30.7) the abscissa x_1 of the weight at the instant corresponding to a new stop, i.e., the distance travelled by the weight during the time t_1

$$x_1 = v_0 t_1 - \frac{v_0}{p} \sin pt_1 + \frac{R_1 - R_2}{k}(1 - \cos pt_1) = v_0 t_1 + \frac{2\alpha v_0}{p}.$$

Taking into account expressions (30.11), we find from formula (30.6) the force of the compressed spring at the time of stopping

$$P(t_1) = 2R_2 - R_1.$$

Since $R_2 < R_1$, $P(t_1) < R_1$; consequently, after stopping the weight will remain stationary for some time until the spring force again reaches the value of the ultimate static-friction force R_1.

During the time the weight is at rest, the force of the compressed spring increases gradually by an amount

$$\Delta P = R_1 - P(t_1) = 2(R_1 - R_2)$$

and the corresponding contraction of the spring is given by

$$\Delta l = \frac{\Delta P}{k} = \frac{2(R_1 - R_2)}{k}.$$

This is equal to the distance travelled by the driving member during the time the weight is at rest. Consequently, the time of rest is

$$t_2 = \frac{\Delta l}{v_0} = \frac{2(R_1 - R_2)}{kv_0} = \frac{2\alpha}{p}. \tag{30.12}$$

The same result can be found from the condition

$$v_0(t_1 + t_2) = x_1$$

which expresses the equality of displacements of the weight and driving member per cycle.

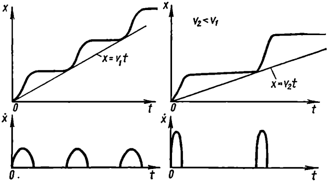


Fig. 159

Thus, the period of self-excited vibration of the weight is given by the formula

$$T = t_1 + t_2.$$

To use this formula, it is necessary first to find t_1 from expression (30.11) and then t_2 from formula (30.12). At the time $t = T$ the next starting of the weight occurs and a new cycle of self-excited vibration begins.

The smaller the velocity of the driving member, the more pronounced is the process of self-excited vibration. Indeed, for small values of v_0 the dimensionless parameter α becomes very large and from expression (30.11) we have approximately in this case

$$\sin pt_1 \to 0, \quad t_1 = \frac{\pi}{p}.$$

Taking into account expression (30.12), the period of self-excited vibration is approximated by

$$T=\frac{\pi+2\alpha}{p}.$$

The value of the second term in the numerator increases with decreasing velocity v_0; the type of motion for two different small values of v_0 is shown in Fig. 159. As the velocity v_0 is reduced, the period of self-excited vibration T increases and so does the value of x_1.

Example 33. Determine the value of the parameter α for which the duration of dwelling of the weight at rest is equal to the duration of its motion. Find the corresponding velocity v_0 if $p=50$ sec^{-1} and $\Delta f=0.05$.

According to the conditions of the problem $t_2=t_1$; hence, denoting

$$pt_1=pt_2=\beta,$$

we have from expressions (30.11) and (30.12)

$$\sin 2\beta=-\frac{2\beta}{1+\beta^2}.$$

The smallest non-zero root of this transcendental equation is $\beta=2.03$. From formula (30.9) we can now obtain the corresponding average velocity

$$v_0=\frac{g}{\alpha p}\Delta f=0.48 \text{ cm/sec}.$$

If the average velocity $v_0<0.48$ cm/sec, the intervals of dwelling at rest will be shorter than the intervals of motion.

As can be seen, the stiffer the system (i.e., the higher its natural frequency p), the smaller is the velocity v_0 corresponding to the conditions of the problem.

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